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Rational transverse shear deformation higher-order theory of anisotropic laminated plates and shells

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Abstract

A rational transverse shear deformation higher-order theory of multilayered anisotropic plates and shallow shells is developed for the solution of statical problems for two possible cases: cross-ply and angle-ply laminates. The theory developed differs from existing ones by three features. Firstly, it is based on the hypotheses which are fully tied to the physical and mechanical characteristics of the anisotropic layers. Secondly, the theory is built on a rational level of difficulty, i.e. it does not add complexity in comparison with other known theories developed for more simple laminated structure. Thirdly, the hypotheses take directly into account the influence of external subject to both normal and tangential loads.

Relying on the specific approach for the derivation of hypotheses all the relations of the stress–strain state of anisotropic laminated shells are obtained. Using the variational approach the system of governing differential equations and corresponding boundary conditions are derived.

The analytical solution for this system is given, and both special cases are stated, namely, cross-ply and angle-ply laminates, for which such solution exists. The results of the calculations are given and compared with exact three-dimensional and some approximate solutions available in the literature. The influence of the laminated structure upon the exactness of results and the characteristics of stress–strain state is studied and discussed.

Keywords: Composite materials; Higher-order theory; Cross-ply and angle-ply laminates

1. Introduction

The investigation of anisotropic shells dates back to 1920s and the first recorded paper on this subject was published by Shtayerman (1924). The use of anisotropic materials in the aircraft construction, later in the rocket production and in many other engineering applications necessitated extensive studies and gave

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impetus to many publications in this field. The basic works by Ambartsumyan (1948, 1961, 1974) and Lekhnitskii (1963, 1968) substantially contributed to the development of the theory of anisotropic plates and shells. These works have already given a considerable attention to the study of the multilayered systems. The list of references is not intended to be a comprehensive one and the specific publications are referred to because of their relevance to the present paper. We will also quote here some monographs dedicated to the anisotropic and laminated composite plates and shells: Riabov (1968), Grigolyuk and Chulkov (1973), Grigorenko (1973), Grigorenko and Vasilenko (1981), Grigorenko et al. (1985), Rikards and Teters (1974), Librescu (1975), Christensen (1979), Bolotin and Novichkov (1980), Piskunov and Verijenko (1986), Piskunov et al. (1987), Rasskazov et al. (1986), Bogdanovich (1987), Whitney (1987), Vasilyev (1988).

Significant sources of information on different theoretical and computational models of the anisotropic and laminated structures are the detailed surveys written by Grygolyuk and Kogan (1972), Grygolyuk and Selezov (1972), Dudchenko et al. (1983), Noor and Burton (1989, 1990a), Noor et al. (1996), Reddy (1990), Reddy and Robbins (1994) and Altenbuch (1998).

Most of the above mentioned monographs on this subject are based on the classical laminated shell theory, incorporating the Kirchoff–Love hypotheses through the entire thickness. It is well known that, due to the anisotropy and heterogeneity of the materials of different layers and the existence of layers which exhibit weak resistance to transverse shear and normal deformations, the classical theory of plates and shells, based on the Kirchoff–Love hypotheses, leads to substantial errors. The possibility of using a three-dimensional (3D) theory is of limited use due to mathematical difficulties and the complexity of the laminated systems.

As a result, numerous theories of plates and shells have been formulated in recent years which approximate the 3D solutions with reasonable accuracy. Such theories have been referred to as non-classical, refined, and higher-order theories and others.

The beginning of the development of refined models links with the name of Timoshenko (1922) who took into account the influence of the transverse shear deformations in the transverse vibrations of bars, and Reissner (1945) for isotropic plates. Later on the main ideas developed in these works became useful outside the scope of their original purpose, have gained recognition and were embodied in special terms like "Timoshenko model" and "Boundary effect of Reissner".

The abandonment of Kirchoff–Love hypothesis (hypothesis of straight normal) and the use of the hypothesis of straight line allowed to take into account the influence of the transverse shear deformations, generalized through the thickness of the plate or the shell, homogeneous or laminated, as a refined factor. When this approach is compared with that of exact 3D solution of the theory of elasticity, it becomes apparent that this factor allowed significantly to refine the normal displacements of the coordinate surface of the structure. However, the hypothesis did not refine the normal stresses which act parallel with this surface since their law of change through the thickness remained linear.

The next step was the derivation of models based on the hypotheses where the distribution law of the transverse stresses and shear deformations occurs according to the quadratic parabola law. Consequently, for the displacements and normal stresses in tangential directions, with non-linear cubic distribution through the thickness were obtained. The final results appear to be in good agreement with the 3D solution.

One widely employed way to build hypotheses of the refined theories is to obtain the distribution law for the transverse shear stresses through the thickness of the layers by means of integration of the equilibrium equations of the 3D theory of the shells when the tangential normal stresses are given. For the anisotropic shells these laws become rather cumbersome. Usually they are simplified by maintaining only the main physical and mechanical characteristics of the layers directly associated with the deformations in the orthogonal directions. This is equivalent to the case when the hypotheses are derived by integration of the equilibrium equations for the cylindrical bending in each orthogonal direction. As this takes place the relations between the hypotheses and the physical and mechanical characteristics of the anisotropic material are considerably violated. The conservation of such relations in full measure leads to irrational complications of the theoretical model when the number of sought functions is increased considerably.

In this study the transverse shear model and the theory developed for it, oriented to solve anisotropic laminated plates and shells are built on a rational approach where the relations are maintained between the hypotheses for transverse shear stresses and the material characteristics of anisotropic layers. The theory developed is given the name "rational transverse shear deformation higher-order theory of anisotropic laminated plates and shells" (RTL).

2. Basic assumptions and classical model

2.1. Basic assumptions

We consider shells with anisotropic layers and which have one surface of elastic symmetry. The shell is represented by a curvilinear orthogonal coordinate system x_1Ox_2 which is parallel to the bounding surfaces and surfaces of contact between the layers (Fig. 1). The axes of the curvilinear coordinates $x_i = \text{const}$ $(i = 1, 2)$ coincide with the principal lines of curvature. The coordinate $x_3 = z$ defined along the normal to the surfaces of the elastic symmetry of the layers and, accordingly, to the reference surface x_1Ox_2 which is positioned arbitrarily through the thickness of the shell. No limitations are placed on the thickness, rigidity, number and/or sequence of the layers. The assumptions that the layers are perfectly bonded ensures their deformation as a single unit without delamination. Thus, the structure of the shell through the thickness can be arbitrarily defined and is heterogeneous.

It is assumed that the coefficients of the first quadratic form of a surface are close to unity, i.e. $A \approx 1$, and the main curvatures are constant, i.e. $k_{ij} = \text{const}$ $(i, j = 1, 2)$. The total thickness of the shell is small in comparison to the radii of the curvatures $(1 + k_{ij} \approx 1)$. These assumptions determine the areas of application of the proposed RTL theory.

Loads are applied on the outer and inner surfaces of the laminate so that

$$p_s^{\pm}(x_i) = p_s^{\pm}, \quad s = 1, 2, 3 \tag{1}$$

where p_s^+ and p_s^- are loads applied on the outer and inner surfaces, respectively, and the subscript s denotes the corresponding coordinate axis. Consequently, the stress conditions on the external surfaces take the following form

$$\sigma_{s3}^{(1)} = -p_s^- \quad \text{for } z = a_0 \;\; (k = 1) \tag{2}$$

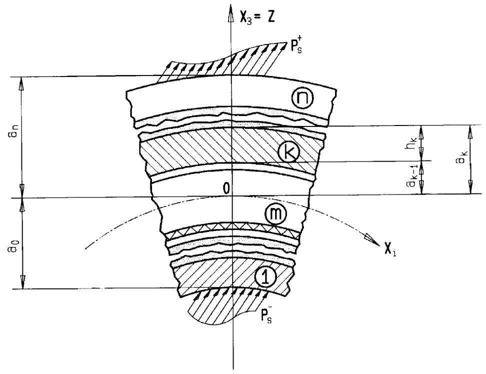

Fig. 1. Geometry of a laminated shell.

$$\sigma_{s3}^{(n)} = +p_s^+ \quad \text{for } z = a_n \ \ (k = n) \ \ s = 1, 2, 3 \tag{3}$$

where k denotes the layer number and n is the total number of layers. Since the layers are assumed to be perfectly bonded, the continuity conditions for an arbitrary surface $z = a_{k-1}$ are given by

$$\sigma_{s3}^{(k)} = \sigma_{s3}^{(k-1)} \quad \text{(static)} \tag{4}$$

$$u_s^{(k)} = u_s^{(k-1)} \quad \text{(kinematic)} \tag{5}$$

In the following derivations, summation is assumed over subscripts i, $j = 1$, 2; s, $r = 1$, 2, 3, and p, q, f, g. However no summation is implied over the index $k = 1, 2, \ldots, m, \ldots, n$. A subscript after a comma denotes differentiation with respect to the variable following the comma and a superscript is expressed in brackets to distinguish it from an exponent.

Considering "small" bending the strain components of the kth layer of the shell may be expressed as

$$\begin{aligned} 2e_{ij}^{(k)} &= u_{i,j}^{(k)} + u_{j,i}^{(k)} + 2k_{ij}u_3^{(k)} \\ 2e_{i3}^{(k)} &= u_{i,3}^{(k)} + u_{3,i}^{(k)} \\ e_{33}^{(k)} &= u_{3,3}^{(k)} \end{aligned} \tag{6}$$

where $u_i^{(k)}(x_i, z)$ and $u_3^{(k)}(x_i, z)$ are displacements of the kth layer in the tangential x_i $(i = 1, 2)$ and normal $z = x_3$ directions, respectively, and k_{ij}s are curvatures of the shell. The displacements of the reference surface $(z = 0, k = m)$ may be expressed as

$$u_i^{(m)}(x_i, 0) = u_i; \qquad u_3^{(m)}(x_i, 0) = w \tag{7}$$

and its deformations and curvature must satisfy the following relations

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) + k_{ij}w; \qquad \kappa_{ij} = -w_{,ij} \tag{8}$$

$$\begin{aligned} &2\varepsilon_{12,12} - \varepsilon_{11,22} - \varepsilon_{22,11} = k_{11}\kappa_{22} + k_{22}\kappa_{11} - 2k_{12}\kappa_{12} \\ &\kappa_{11,2} - \kappa_{12,1} = 0; \qquad \kappa_{22,1} - \kappa_{12,2} = 0 \end{aligned} \tag{9}$$

The generalized Hooke's law for an anisotropic layer k of the shell, where the surface of elastic symmetry at any point (x_i, z) is orthogonal to the normal, may be expressed as (Ambartsumyan, 1974)

$$\begin{aligned} e_{11}^{(k)} &= a_{11}^{(k)}\sigma_{11}^{(k)} + a_{12}^{(k)}\sigma_{22}^{(k)} + a_{13}^{(k)}\sigma_{33}^{(k)} + a_{16}^{(k)}\sigma_{12}^{(k)} \\ e_{22}^{(k)} &= a_{21}^{(k)}\sigma_{11}^{(k)} + a_{22}^{(k)}\sigma_{22}^{(k)} + a_{23}^{(k)}\sigma_{33}^{(k)} + a_{26}^{(k)}\sigma_{12}^{(k)} \\ e_{33}^{(k)} &= a_{31}^{(k)}\sigma_{11}^{(k)} + a_{32}^{(k)}\sigma_{22}^{(k)} + a_{33}^{(k)}\sigma_{33}^{(k)} + a_{36}^{(k)}\sigma_{12}^{(k)} \\ 2e_{23}^{(k)} &= a_{44}^{(k)}\sigma_{23}^{(k)} + a_{45}^{(k)}\sigma_{13}^{(k)} \\ 2e_{13}^{(k)} &= a_{54}^{(k)}\sigma_{23}^{(k)} + a_{55}^{(k)}\sigma_{13}^{(k)} \\ 2e_{12}^{(k)} &= a_{61}^{(k)}\sigma_{11}^{(k)} + a_{62}^{(k)}\sigma_{22}^{(k)} + a_{63}^{(k)}\sigma_{33}^{(k)} + a_{66}^{(k)}\sigma_{12}^{(k)} \end{aligned} \tag{10}$$

where $a_{11}^{(k)}, a_{12}^{(k)}, \ldots, a_{66}^{(k)}$ are the elastic compliance coefficients of the kth layer. Correspondingly, the stresses can be expressed in terms of the deformations

$$\begin{aligned}
\sigma_{11}^{(k)} &= A_{11}^{(k)}e_{11}^{(k)} + A_{12}^{(k)}e_{22}^{(k)} + A_{13}^{(k)}e_{33}^{(k)} + 2A_{16}^{(k)}e_{12}^{(k)} \\
\sigma_{22}^{(k)} &= A_{21}^{(k)}e_{11}^{(k)} + A_{22}^{(k)}e_{22}^{(k)} + A_{23}^{(k)}e_{33}^{(k)} + 2A_{26}^{(k)}e_{12}^{(k)} \\
\sigma_{33}^{(k)} &= A_{31}^{(k)}e_{11}^{(k)} + A_{32}^{(k)}e_{22}^{(k)} + A_{33}^{(k)}e_{33}^{(k)} + 2A_{36}^{(k)}e_{12}^{(k)} \\
\sigma_{23}^{(k)} &= 2A_{44}^{(k)}e_{23}^{(k)} + 2A_{45}^{(k)}e_{13}^{(k)} \\
\sigma_{13}^{(k)} &= 2A_{54}^{(k)}e_{23}^{(k)} + 2A_{55}^{(k)}e_{13}^{(k)} \\
\sigma_{12}^{(k)} &= A_{61}^{(k)}e_{11}^{(k)} + A_{62}^{(k)}e_{22}^{(k)} + A_{63}^{(k)}e_{33}^{(k)} + 2A_{66}^{(k)}e_{12}^{(k)}
\end{aligned} \tag{11}$$

where $A_{11}^{(k)}, A_{22}^{(k)}, \ldots, A_{66}^{(k)}$ are stiffness parameters of the kth layer.

The geometrical equations (6) and the physical Hooke's law (10) and (11) predetermine the geometrical and physical linearity of RTL. The derivation of the non-linear RTL of the anisotropic laminated plates and shells is a specific problem of the future. For some special cases of multilayered orthotropic shells a non-linear theory was developed in (Verijenko, 1989, 1994).

2.2. Classical model

We assume that compliance coefficients for the shell material in the transversal direction are excluded, i.e.

$$\begin{aligned}
&a_{13}^{(k)} = a_{31}^{(k)} = 0; \qquad a_{23}^{(k)} = a_{32}^{(k)} = 0 \\
&a_{33}^{(k)} = 0; \qquad a_{36}^{(k)} = a_{63}^{(k)} = 0 \\
&a_{44}^{(k)} = a_{55}^{(k)} = a_{45}^{(k)} = a_{54}^{(k)} = 0
\end{aligned} \tag{12}$$

In this case the kinematic Kirchoff–Love hypotheses

$$2e_{i3}^{(k)} = 0; \quad e_{33}^{(k)} = 0; \qquad i = 1, 2 \tag{13}$$

follow from the Hooke's law. It should be noted that the use of the hypothesis (13) in the general Hooke's law (11) allows to obtain the following expressions

$$\begin{aligned}
\sigma_{11}^{(k)} &= A_{11}^{(k)}e_{11}^{(k)} + A_{12}^{(k)}e_{22}^{(k)} + 2A_{16}^{(k)}e_{12}^{(k)} \\
\sigma_{22}^{(k)} &= A_{21}^{(k)}e_{11}^{(k)} + A_{22}^{(k)}e_{22}^{(k)} + 2A_{26}^{(k)}e_{12}^{(k)} \\
\sigma_{12}^{(k)} &= A_{16}^{(k)}e_{11}^{(k)} + A_{26}^{(k)}e_{22}^{(k)} + 2A_{66}^{(k)}e_{12}^{(k)}
\end{aligned} \tag{14}$$

and also an equality to zero of the transverse shear stresses

$$\sigma_{13}^{(k)} = \sigma_{23}^{(k)} = 0 \tag{15}$$

However, an equality to zero of the transverse normal stresses, which are usually neglected in the classical theory, does not follow from Eq. (13). That is why we specially introduce the statical Kirchoff–Love hypothesis

$$\sigma_{33}^{(k)} = 0 \tag{16}$$

Using the hypotheses (13) in the second and third expressions in Eq. (6) and, integrating them, we obtain the classical kinematic model of the shell, displacements in the kth layer

$$u_i^{(k)} = u_i - w_{,i}z; \qquad u_3^{(k)} = w \tag{17}$$

These relations do not contradict the conditions (5). Then substituting Eq. (17) into Eqs. (6)–(8) the deformations in the tangential directions for the kth layer may be obtained as

$$e_{ij}^{(k)} = \varepsilon_{ij} + \kappa_{ij} z, \quad i,j = 1,2 \tag{18}$$

where $e_{ij}^{(k)} = e_{ji}^{(k)}$, $\varepsilon_{ij} = \varepsilon_{ji}$, $\kappa_{ij} = \kappa_{ji}$. After substituting Eq. (18) into Eq. (14) we obtain the normal stresses in the tangential directions expressed in terms of the deformations of the coordinate surface.

$$\begin{aligned}
\sigma_{11}^{(k)} &= A_{11}^{(k)}\varepsilon_{11} + A_{12}^{(k)}\varepsilon_{22} + 2A_{16}^{(k)}\varepsilon_{12} + (A_{11}^{(k)}\kappa_{11} + A_{12}^{(k)}\kappa_{22} + 2A_{16}^{(k)}\kappa_{12})z \\
\sigma_{22}^{(k)} &= A_{21}^{(k)}\varepsilon_{11} + A_{22}^{(k)}\varepsilon_{22} + 2A_{26}^{(k)}\varepsilon_{12} + (A_{21}^{(k)}\kappa_{11} + A_{22}^{(k)}\kappa_{22} + 2A_{26}^{(k)}\kappa_{12})z \\
\sigma_{12}^{(k)} &= A_{16}^{(k)}\varepsilon_{11} + A_{26}^{(k)}\varepsilon_{22} + 2A_{66}^{(k)}\varepsilon_{12} + (A_{16}^{(k)}\kappa_{11} + A_{26}^{(k)}\kappa_{22} + 2A_{66}^{(k)}\kappa_{12})z
\end{aligned} \tag{19}$$

The transversal terms of the stress tensor, according to Eqs. (15) and (16) are equal to zero. However, they can be determined from the equilibrium equations of the 3D theory of elasticity of the shells. In these expressions we consider the tangential terms to be known and expressed in accordance with Eq. (19). These equations (Novozhilov, 1962) we will write for the kth layer of the shell as

$$\sigma_{ij,j}^{(k)} + \sigma_{i3,3}^{(k)} = 0; \qquad \sigma_{33,3}^{(k)} + \sigma_{i3,3}^{(k)} - k_{ij}\sigma_{ij}^{(k)} = 0; \quad i,j = 1,2 \tag{20}$$

From these equations and condition (2) on the inner surface of the shell $z = a_0$ we obtain the transverse shear and normal stresses for the kth layer

$$\sigma_{i3}^{(k)} = -p_i^- - \int_{a_0}^{z} \sigma_{ij,j}^{(k)}\,\mathrm{d}z, \quad i = 1,2 \tag{21}$$

$$\sigma_{33}^{(k)} = -p_3^- - \int_{a_0}^{z} (\sigma_{i3,i}^{(k)} - k_{ij}\sigma_{ij}^{(k)})\,\mathrm{d}z \tag{22}$$

The expressions for the outer surface of the shell $z = a_n$, using conditions (3) and (4), may be given as

$$p_i^- + p_i^+ = \int_{a_0}^{a_n} \sigma_{ij,j}^{(k)}\,\mathrm{d}z, \quad i,j = 1,2 \tag{23}$$

$$p_3^- + p_3^+ = \int_{a_0}^{a_n} (\sigma_{i3,3}^{(k)} - k_{ij}\sigma_{ij}^{(k)})\,\mathrm{d}z \tag{24}$$

which are the equilibrium equations of the shell in the projections on the axes x_i and $x_3 = z$, respectively.

Next we introduce the integral characteristics of the stresses, namely the internal forces and moments

$$[N_{ij}, M_{ij}] = \int_{a_0}^{a_n} [\sigma_{ij}^{(k)}, \sigma_{ij}^{(k)} z]\,\mathrm{d}z, \quad i,j = 1,2 \tag{25}$$

Then the equilibrium equations (23) and (24) can be transformed into

$$\begin{aligned}
&N_{ij,j} + (p_i^+ + p_i^-) = 0 \\
&M_{ij,ij} - k_{ij}N_{ij} + (p_3^+ + p_3^-) + (p_{i,i}^+ a_n + p_{i,i}^- a_0) = 0 \quad i,j = 1,2
\end{aligned} \tag{26}$$

Eq. (26) constitute the system of equations of the classical theory of the multilayered anisotropic shallow shells. In the following, this theory will be considered as a special case for the developing non-classical theory.

3. Transverse shear stresses and strains

3.1. General form of transverse shear stresses

The expression (21) gives the transverse shear stresses in general form. In order to solve specific problems of the stress–strain state of the shell it is necessary to derive expanded expressions. They are also essential for later use as an integral part of obtaining the hypotheses of non-classical theory of the highest approximation.

Next we substitute the relation (19) into the expressions (21) and (23). Considering the case when $i = 1$ we obtain the following expressions

$$\begin{aligned}\sigma_{13}^{(k)} &= -p_1^- - \int_{a_0}^{z} (\sigma_{11,1}^{(k)} + \sigma_{12,2}^{(k)})\,\mathrm{d}z \\ &= -p_1^- - \left[\varepsilon_{11,1}\int_{a_0}^{z} A_{11}^{(k)}\,\mathrm{d}z + \varepsilon_{22,1}\int_{a_0}^{z} A_{12}^{(k)}\,\mathrm{d}z + (2\varepsilon_{12,1}+\varepsilon_{11,2})\int_{a_0}^{z} A_{16}^{(k)}\,\mathrm{d}z + \varepsilon_{22,2}\int_{a_0}^{z} A_{26}^{(k)}\,\mathrm{d}z \right.\\ &\quad \left. + 2\varepsilon_{12,2}\int_{a_0}^{z} A_{66}^{(k)}\,\mathrm{d}z\right] - \left[\kappa_{11,1}\int_{a_0}^{z} A_{11}^{(k)} z\,\mathrm{d}z + \kappa_{22,1}\int_{a_0}^{z} A_{12}^{(k)} z\,\mathrm{d}z + (2\kappa_{12,1}+\kappa_{11,2})\int_{a_0}^{z} A_{16}^{(k)} z\,\mathrm{d}z \right.\\ &\quad \left. + \kappa_{22,2}\int_{a_0}^{z} A_{26}^{(k)} z\,\mathrm{d}z + 2\kappa_{12,2}\int_{a_0}^{z} A_{66}^{(k)} z\,\mathrm{d}z\right]\end{aligned} \tag{27}$$

$$\begin{aligned}p_1^- + p_1^+ = &-\left[\varepsilon_{11,1}\int_{a_0}^{a_n} A_{11}^{(k)}\,\mathrm{d}z + \varepsilon_{22,1}\int_{a_0}^{a_n} A_{12}^{(k)}\,\mathrm{d}z + (2\varepsilon_{12,1}+\varepsilon_{11,2})\int_{a_0}^{a_n} A_{16}^{(k)}\,\mathrm{d}z + \varepsilon_{22,2}\int_{a_0}^{a_n} A_{26}^{(k)}\,\mathrm{d}z \right.\\ &\left. + 2\varepsilon_{12,2}\int_{a_0}^{a_n} A_{66}^{(k)} z\,\mathrm{d}z\right] - \left[\kappa_{11,1}\int_{a_0}^{a_n} A_{11}^{(k)} z\,\mathrm{d}z + \kappa_{22,1}\int_{a_0}^{a_n} A_{12}^{(k)} z\,\mathrm{d}z + (2\kappa_{12,1} \right.\\ &\left. + \kappa_{11,2})\int_{a_0}^{a_n} A_{16} z\,\mathrm{d}z + \kappa_{22,2}\int_{a_0}^{a_n} A_{26}^{(k)} z\,\mathrm{d}z + 2\kappa_{12,2}\int_{a_0}^{a_n} A_{66}^{(k)} z\,\mathrm{d}z\right]\end{aligned} \tag{28}$$

Let us introduce the following distribution functions of the stress terms through the thickness

$$\begin{aligned}\left[f_{i1}^{(k)};\ f_{i2}^{(k)};\ f_{i6}^{(k)};\ f_{66}^{(k)}\right] &= \int_{a_0}^{z} \left[A_{i1}^{(k)};\ A_{i2}^{(k)};\ A_{i6}^{(k)};\ A_{66}^{(k)}\right]\mathrm{d}z \\ \left[F_{i1}^{(k)};\ F_{i2}^{(k)};\ F_{i6}^{(k)};\ F_{66}^{(k)}\right] &= \int_{a_0}^{z} \left[A_{i1}^{(k)};\ A_{i2}^{(k)};\ A_{i6}^{(k)};\ A_{66}^{(k)}\right] z\,\mathrm{d}z\end{aligned} \tag{29}$$

The following constants, characteristics of the layer rigidity, correspond to these functions

$$\begin{aligned}[B_{i1};\ B_{i2};\ B_{i6};\ B_{66}] &= \int_{a_0}^{a_n} \left[A_{i1}^{(k)};\ A_{i2}^{(k)};\ A_{i6}^{(k)};\ A_{66}^{(k)}\right]\mathrm{d}z \\ [C_{i1};\ C_{i2};\ C_{i6};\ C_{66}] &= \int_{a_0}^{a_n} \left[A_{i1}^{(k)};\ A_{i2}^{(k)};\ A_{i6}^{(k)};\ A_{66}^{(k)}\right] z\,\mathrm{d}z\end{aligned} \tag{30}$$

In Eqs. (29) and (30) we assume that $i = 1, 2$. From expression (28) we can derive any term of the deformation, for example

$$\begin{aligned}\varepsilon_{11,1} = &-\frac{1}{B_{11}}\left[(p_1^- + p_1^+) + \varepsilon_{22,1}B_{12} + (2\varepsilon_{12,1}+\varepsilon_{11,2})B_{16} + \varepsilon_{22,2}B_{26} + 2\varepsilon_{12,2}B_{66} + \kappa_{11,1}C_{11} + \kappa_{22,1}C_{12} \right.\\ &\left. + (2\kappa_{12,1}+\varepsilon_{22,1})C_{16} + \kappa_{22,2}C_{26} + 2\kappa_{12,2}C_{66}\right]\end{aligned} \tag{31}$$

Now substituting Eq. (31) into Eq. (27) the final expression may be obtained

$$\sigma_{13}^{(k)} = p_1^+ \frac{f_{11}^{(k)}}{B_{11}} + p_1^- \left(\frac{f_{11}^{(k)}}{B_{11}} - 1 \right) - \left[\varepsilon_{22,1} \left(f_{12}^{(k)} - \frac{B_{12}}{B_{11}} f_{11}^{(k)} \right) + (2\varepsilon_{12,1} + \varepsilon_{11,2}) \left(f_{16}^{(k)} - \frac{B_{16}}{B_{11}} f_{11}^{(k)} \right) \right.$$
$$+ \varepsilon_{22,2} \left(f_{26}^{(k)} - \frac{B_{26}}{B_{11}} f_{11}^{(k)} \right) + 2\varepsilon_{12,2} \left(f_{66}^{(k)} - \frac{B_{66}}{B_{11}} f_{11}^{(k)} \right) + \kappa_{11,1} \left(F_{11}^{(k)} - \frac{C_{11}}{B_{11}} f_{11}^{(k)} \right) + \kappa_{22,1} \left(F_{12}^{(k)} - \frac{C_{12}}{B_{11}} f_{11}^{(k)} \right)$$
$$\left. + (2\kappa_{12,1} + \kappa_{11,2}) \left(F_{16}^{(k)} - \frac{C_{16}}{B_{11}} f_{11}^{(k)} \right) + \kappa_{22,2} \left(F_{26}^{(k)} - \frac{C_{26}}{B_{11}} f_{11}^{(k)} \right) + 2\kappa_{12,2} \left(F_{66}^{(k)} - \frac{C_{66}}{B_{11}} f_{11}^{(k)} \right) \right] \quad (32)$$

The stresses $\sigma_{23}^{(k)}$ can be obtained in a similar manner. The distinctive feature of the expressions for the general form of the transverse shear stresses is that they satisfy the loading conditions on the external surfaces and the interface conditions on the contact surfaces of the layers when the coordinate surface is positioned arbitrary through the thickness of layers.

3.2. Rational form of transverse shear stresses

In the derivation of the non-classical theory the influence of the transverse shear deformations and corresponding transverse shear stresses must be taken into account. The expressions for the transverse shear stresses may be used as a basis for the hypotheses of this theory. They fully reflect the physical and mechanical characteristics of the anisotropic layers. However, each term gives rise to new sought functions. Their number appears to be unnecessarily large and leads to the creation of a complicated theory. Nevertheless, a "rational arbitrary rule" may be employed to simplify them and make the theory rational. For the purpose of simplification and without changing the principles of the theory, we will represent the expression (27) for the transverse shear stresses as a sum of items, viz

$$\sigma_{13}^{(k)} = \sum_{m=1}^{5} \sigma_{13m}^{(k)} \quad (33)$$

Each item here represents a part of the stress which is associated with some rigidity characteristics of the material for the layer k. We will connect the tangential load p_1^- with items which contain the deformations of the coordinate surface arising directly in its direction. The above mentioned items can be written as

$$\begin{aligned}
\sigma_{131}^{(k)} &= -p_1^- - (\varepsilon_{11,1} f_{11}^{(k)} + \kappa_{11,1} F_{11}^{(k)}) \\
\sigma_{132}^{(k)} &= -(\varepsilon_{22,1} f_{12}^{(k)} + \kappa_{22,1} F_{12}^{(k)}) \\
\sigma_{133}^{(k)} &= -\left[(\varepsilon_{11,2} + 2\varepsilon_{12,1}) f_{16}^{(k)} + (\kappa_{11,2} + 2\kappa_{12,1}) F_{16}^{(k)} \right] \\
\sigma_{134}^{(k)} &= -(\varepsilon_{22,2} f_{26}^{(k)} + \kappa_{22,2} F_{26}^{(k)}) \\
\sigma_{135}^{(k)} &= -2(\varepsilon_{12,2} f_{66}^{(k)} + \kappa_{12,2} F_{66}^{(k)})
\end{aligned} \quad (34)$$

In accordance with the representation of the stress $\sigma_{13}^{(k)}$ as a sum of five items we replace the equilibrium equation (28) with five relations, which taken together satisfy its conditions. They are

$$\begin{aligned}
& p_1^- + p_1^+ = -(\varepsilon_{11,1} B_{11} + \kappa_{11,1} C_{11}) \\
& \varepsilon_{22,1} B_{12} + \kappa_{22,1} C_{12} = 0 \\
& (\varepsilon_{11,2} + 2\varepsilon_{12,1}) B_{16} + (\kappa_{11,2} + 2\kappa_{12,1}) C_{16} = 0 \\
& \varepsilon_{22,2} B_{26} + \kappa_{22,2} C_{26} = 0 \\
& \varepsilon_{12,2} B_{66} + \kappa_{12,2} C_{66} = 0
\end{aligned} \quad (35)$$

From Eq. (35) the following expressions can be found

$$\begin{aligned}
&\varepsilon_{11,1} = -\left(\frac{p_1^- + p_1^+}{B_{11}} + \kappa_{11,1}\frac{C_{11}}{B_{11}}\right) \\
&\varepsilon_{22,1} = -\kappa_{22,1}\frac{C_{12}}{B_{12}} \\
&(\varepsilon_{11,2} + 2\varepsilon_{12,1}) = -(\kappa_{11,2} + 2\kappa_{12,1})\frac{C_{16}}{B_{16}} \\
&\varepsilon_{22,2} = -\kappa_{22,2}\frac{C_{26}}{B_{26}}; \qquad \varepsilon_{12,2} = -\kappa_{12,2}\frac{C_{66}}{B_{66}}
\end{aligned} \tag{36}$$

Substituting them in Eq. (34) and summing according to Eq. (33) we obtain the following expressions for the transverse shear stresses for the kth layer of the shell

$$\begin{aligned}
\sigma_{13}^{(k)} = {} & p_1^+\frac{f_{11}^{(k)}}{B_{11}} + p_1^-\left(\frac{f_{11}^{(k)}}{B_{11}} - 1\right) - \kappa_{11,1}\left(F_{11}^{(k)} - \frac{C_{11}}{B_{11}}f_{11}^{(k)}\right) - \kappa_{22,1}\left[\left(F_{12}^{(k)} + 2F_{66}^{(k)}\right) - \left(\frac{C_{12}}{B_{12}}f_{12}^{(k)} + 2\frac{C_{66}}{B_{66}}f_{66}^{(k)}\right)\right] \\
& - 3\kappa_{11,2}\left(F_{16}^{(k)} - \frac{C_{16}}{B_{16}}f_{16}^{(k)}\right) - \kappa_{22,2}\left(F_{26}^{(k)} - \frac{C_{26}}{B_{26}}f_{26}^{(k)}\right)
\end{aligned} \tag{37}$$

In a similar manner we have

$$\begin{aligned}
\sigma_{23}^{(k)} = {} & p_2^+\frac{f_{22}^{(k)}}{B_{22}} + p_2^-\left(\frac{f_{22}^{(k)}}{B_{22}} - 1\right) - \kappa_{22,2}\left(F_{22}^{(k)} - \frac{C_{22}}{B_{22}}f_{22}^{(k)}\right) - \kappa_{11,2}\left[\left(F_{21}^{(k)} + 2F_{66}^{(k)}\right) - \left(\frac{C_{21}}{B_{21}}f_{21}^{(k)} + 2\frac{C_{66}}{B_{66}}f_{66}^{(k)}\right)\right] \\
& - 3\kappa_{22,1}\left(F_{26}^{(k)} - \frac{C_{26}}{B_{26}}f_{26}^{(k)}\right) - \kappa_{11,1}\left(F_{16}^{(k)} - \frac{C_{16}}{B_{16}}f_{16}^{(k)}\right)
\end{aligned} \tag{38}$$

Next we introduce the following distribution functions of the transverse shear stresses through the thickness of the laminate

$$\begin{aligned}
&\varphi_{1k}^+ = \frac{f_{11}^{(k)}}{B_{11}}; \qquad \varphi_{2k}^+ = \frac{f_{22}^{(k)}}{B_{22}} \\
&\varphi_{1k}^- = \frac{f_{11}^{(k)}}{B_{11}} - 1; \qquad \varphi_{2k}^- = \frac{f_{22}^{(k)}}{B_{22}} - 1 \\
&\varphi_{11}^{(k)} = F_{11}^{(k)} - \frac{C_{11}}{B_{11}}f_{11}^{(k)}; \qquad \varphi_{21}^{(k)} = F_{22}^{(k)} - \frac{C_{22}}{B_{22}}f_{22}^{(k)} \\
&\varphi_{12}^{(k)} = (F_{12}^{(k)} + 2F_{66}^{(k)}) - \left(\frac{C_{12}}{B_{12}}f_{12}^{(k)} + 2\frac{C_{66}}{B_{66}}f_{66}^{(k)}\right) \\
&\varphi_{22}^{(k)} = (F_{21}^{(k)} + 2F_{66}^{(k)}) - \left(\frac{C_{21}}{B_{21}}f_{21} + 2\frac{C_{66}}{B_{66}}f_{66}^{(k)}\right) \\
&\varphi_{13}^{(k)} = 3\left(F_{16}^{(k)} - \frac{C_{16}}{B_{16}}f_{16}^{(k)}\right); \qquad \varphi_{23}^{(k)} = 3\left(F_{26}^{(k)} - \frac{C_{26}}{B_{26}}f_{26}^{(k)}\right) \\
&\varphi_{14}^{(k)} = \left(F_{26}^{(k)} - \frac{C_{26}}{B_{26}}f_{26}^{(k)}\right); \qquad \varphi_{24}^{(k)} = \left(F_{16}^{(k)} - \frac{C_{16}}{B_{16}}f_{16}^{(k)}\right)
\end{aligned} \tag{39}$$

Substituting Eq. (39) into Eqs. (37) and (38) the final expressions can be found

$$\begin{aligned}\sigma_{13}^{(k)} &= p_1^- \varphi_{1k}^- + p_1^+ \varphi_{1k}^+ - (\kappa_{11,1}\varphi_{11}^{(k)} + \kappa_{22,1}\varphi_{12}^{(k)} + \kappa_{11,2}\varphi_{13}^{(k)} + \kappa_{22,2}\varphi_{14}^{(k)})\\ \sigma_{23}^{(k)} &= p_2^- \varphi_{2k}^- + p_2^+ \varphi_{2k}^+ - (\kappa_{22,2}\varphi_{21}^{(k)} + \kappa_{11,2}\varphi_{22}^{(k)} + \kappa_{22,1}\varphi_{23}^{(k)} + \kappa_{11,1}\varphi_{24}^{(k)})\end{aligned} \tag{40}$$

The derived expressions do not contain terms with the tangential deformations of the coordinate surface as compared to earlier obtained expressions (32). The influence of the deformations is taken into account indirectly by the distribution functions (39). The expression (40) satisfies both the conditions on the external and intermediate surfaces of the shell. They contain less number of items (6 each instead of 11) and consequently will give rise to a less number of sought functions during the derivation of the non-classical theory. We take them as a basis for the further derivations, first of all, for the determination of the transverse shear strains.

3.3. Transverse shear strains

Using Eq. (40) the transverse shear strains can be obtained from the Hooke's law (10) as

$$\begin{aligned}2e_{13}^{(k)} &= a_{55}^{(k)}\sigma_{13}^{(k)} + a_{54}^{(k)}\sigma_{23}^{(k)}\\ &= -\Big[\kappa_{11,1}(\varphi_{11}^{(k)}a_{55}^{(k)} + \varphi_{24}^{(k)}a_{54}^{(k)}) + \kappa_{22,1}(\varphi_{12}^{(k)}a_{55}^{(k)} + \varphi_{23}^{(k)}a_{54}^{(k)}) + \kappa_{11,2}(\varphi_{13}^{(k)}a_{55}^{(k)} + \varphi_{22}^{(k)}a_{54}^{(k)})\\ &\quad + \kappa_{22,2}(\varphi_{14}^{(k)}a_{55}^{(k)} + \varphi_{21}^{(k)}a_{54}^{(k)})\Big] + (p_1^-\varphi_{1k}^- + p_1^+\varphi_{1k}^+)a_{55}^{(k)} + (p_2^-\varphi_{2k}^- + p_2^+\varphi_{2k}^+)a_{54}^{(k)}\\ 2e_{23}^{(k)} &= a_{44}^{(k)}\sigma_{23}^{(k)} + a_{45}^{(k)}\sigma_{13}^{(k)}\\ &= -\Big[\kappa_{22,2}(\varphi_{21}^{(k)}a_{44}^{(k)} + \varphi_{14}^{(k)}a_{45}^{(k)}) + \kappa_{11,2}(\varphi_{22}^{(k)}a_{44}^{(k)} + \varphi_{13}^{(k)}a_{45}^{(k)}) + \kappa_{22,1}(\varphi_{23}^{(k)}a_{44}^{(k)} + \varphi_{12}^{(k)}a_{45}^{(k)})\\ &\quad + \kappa_{11,1}(\varphi_{24}^{(k)}a_{44}^{(k)} + \varphi_{11}^{(k)}a_{45}^{(k)})\Big] + (p_2^-\varphi_{2k}^- + p_2^+\varphi_{2k}^+)a_{44}^{(k)} + (p_1^-\varphi_{1k}^- + p_1^+\varphi_{1k}^+)a_{45}^{(k)}\end{aligned} \tag{41}$$

In Eq. (41) through the thickness distribution functions of the transverse shear may be defined as

$$\begin{aligned}\Psi_{1r}^{(k)} &= -(\varphi_{1r}^{(k)}a_{55}^{(k)} + \varphi_{2s}^{(k)}a_{54}^{(k)})\\ \Psi_{2r}^{(k)} &= -(\varphi_{2s}^{(k)}a_{44}^{(k)} + \varphi_{1r}^{(k)}a_{45}^{(k)})\\ r,s &= 1,4;2,3;3,2;4,1\end{aligned} \tag{42}$$

$$\begin{aligned}&\Psi_{15}^{(k)} = \varphi_{1k}^- a_{55}; \quad \Psi_{16}^{(k)} = \varphi_{1k}^+ a_{55}; \quad \Psi_{17}^{(k)} = \varphi_{2k}^- a_{54}; \quad \Psi_{18}^{(k)} = \varphi_{2k}^+ a_{54}\\ &\Psi_{25}^{(k)} = \varphi_{1k}^- a_{45}; \quad \Psi_{26}^{(k)} = \varphi_{1k}^+ a_{45}; \quad \Psi_{27}^{(k)} = \varphi_{2k}^- a_{44}; \quad \Psi_{28}^{(k)} = \varphi_{2k}^+ a_{44}\end{aligned} \tag{43}$$

These functions establish the full tie between the transverse shear strains and the physical and mechanical characteristics of the anisotropic layers. Thus Eq. (41) takes the following form

$$\begin{aligned}2e_{13}^{(k)} &= \kappa_{11,1}\Psi_{11}^{(k)} + \kappa_{22,1}\Psi_{12}^{(k)} + \kappa_{11,2}\Psi_{13}^{(k)} + \kappa_{22,2}\Psi_{14}^{(k)} + p_1^-\Psi_{15}^{(k)} + p_1^+\Psi_{16}^{(k)} + p_2^-\Psi_{17}^{(k)} + p_2^+\Psi_{28}^{(k)}\\ 2e_{23}^{(k)} &= \kappa_{11,1}\Psi_{21}^{(k)} + \kappa_{22,1}\Psi_{22}^{(k)} + \kappa_{11,2}\Psi_{23}^{(k)} + \kappa_{22,2}\Psi_{24}^{(k)} + p_1^-\Psi_{25}^{(k)} + p_1^+\Psi_{26}^{(k)} + p_2^-\Psi_{27}^{(k)} + p_2^+\Psi_{28}^{(k)}\end{aligned} \tag{44}$$

The expressions for the transverse shear strains are important for the derivation of the non-classical higher-order theory.

4. Derivation of the non-classical theory

4.1. Hypotheses and displacements

In deriving a non-classical higher-order transverse shear theory we assume that transverse shear strains are not equal to zero, that is

$$2e_{i3}^{(k)} \neq 0; \quad i = 1, 2 \tag{45}$$

However, we assume that the transverse normal deformations and transverse normal stresses, as in the classical theory, are equal to zero

$$e_{33}^{(k)} = 0; \quad \sigma_{33}^{(k)} = 0 \tag{46}$$

We are not changing the hypothesis for the normal strains and it is equal to zero. Therefore, the normal displacements are constants through the thickness of the laminated shell and are equal to those displacements on the reference surface

$$u_3^{(k)} = u_3(x_i, 0) = w \tag{47}$$

Using Eq. (44) for the transverse shear strains and the strain–displacement relations (6), we can find more accurate components of the tangential displacements. From the second expression in Eq. (6) we obtain

$$u_{i,3}^{(k)} = 2e_{i3}^{(k)} - u_{3,i}^{(k)} \tag{48}$$

and after integrating of this relation we have

$$u_i^{(k)} = u_i + \int_0^z (2e_{i3}^{(k)} - u_{3,i}^{(k)})\,\mathrm{d}z \tag{49}$$

We introduce the following distribution functions of the tangential displacements

$$\psi_{ip}^{(k)} = -\int_0^z \Psi_{ip}^{(k)}\,\mathrm{d}z, \quad i = 1, 2; \ p = 1, 2, \ldots, 8 \tag{50}$$

Using these functions the expression for the tangential displacements may be written as

$$\begin{aligned} u_1^{(k)} &= u_1 - (w_{,1}z + \kappa_{11,1}\psi_{11}^{(k)} + \kappa_{22,1}\psi_{12}^{(k)} + \kappa_{11,2}\psi_{13}^{(k)} + \kappa_{22,2}\psi_{14}^{(k)} + p_1^-\psi_{15}^{(k)} + p_1^+\psi_{16}^{(k)} + p_2^-\psi_{17}^{(k)} + p_2^+\psi_{18}^{(k)}) \\ u_2^{(k)} &= u_2 - (w_{,2}z + \kappa_{11,1}\psi_{21}^{(k)} + \kappa_{22,1}\psi_{22}^{(k)} + \kappa_{11,2}\psi_{23}^{(k)} + \kappa_{22,2}\psi_{24}^{(k)} + p_1^-\psi_{25}^{(k)} + p_1^+\psi_{26}^{(k)} + p_2^-\psi_{27}^{(k)} + p_2^+\psi_{28}^{(k)}) \end{aligned} \tag{51}$$

where κ_{ij} can be determined from the second relation in Eq. (8). The distribution functions defined in Eq. (50) allow us to satisfy the continuity conditions in between the layers for the tangential displacements when the reference surface is positioned arbitrary through the thickness of the shell.

4.2. Relations for the non-classical theory

First two terms in the expression (51) contain unknown functions u_1, u_2, w. These items are from the classical theory whereas the rest of items are new. They take into account the influence of the transverse shear strains on the tangential displacements. One can be sure that the highest degree of the polynomials, which results from the distribution function (50) written in an explicit form, equals three. Thus, the tangential displacements are non-linear through the thickness of the shell.

In order to derive relations of the non-classical theory we retain first two terms in the expressions (51). For the next four terms we introduce new unknown functions of the reference surface using the following irreversible relations

$$[\kappa_{11,1};\ \kappa_{22,1};\ \kappa_{11,2};\ \kappa_{22,2}] \rightarrow [\chi_1,;\ \chi_2;\ \chi_3;\ \chi_4] \tag{52}$$

Additionally, we introduce the following relations for the functions of the given external load

$$[p_1^-;\ p_1^+;\ p_2^-;\ p_2^+] = [\chi_5,;\ \chi_6;\ \chi_7;\ \chi_8] \tag{53}$$

Replacing the functions in the relations (51) in accordance with the relationships defined by Eqs. (52) and (53), the expression for the tangential displacements may be written as

$$u_i^{(k)} = u_i - w_{,i}z - \chi_p \psi_{ip}; \quad i = 1,2;\ p = 1,2,\ldots,8 \tag{54}$$

In this expression summation is assumed over "mute" index p. Items containing this index take into account the influence of the transverse shear deformations as a result of the effect of the transverse shear stresses. The last four terms ($p = 5,6,7,8$) account for the direct effect of impact of the external load.

The tangential displacements in the form of Eq. (54) including the expression (47) for the normal displacements represent the non-classical kinematic model of the shell. The classical model may be obtained by specifying the material properties. In this case the compliance coefficients responsible for the shear must be equal to zero, i.e. the last relation (12) is satisfied, and also we have $\psi_{1p} = 0$.

Let us now obtain the components of the strain tensor for the kth layer. Taking into account the kinematic model (47) and (54), the tangential components in Eq. (6) may be written as

$$\begin{aligned} e_{ij}^{(k)} &= \tfrac{1}{2}(u_{i,j}^{(k)} + u_{j,i}^{(k)}) + k_{ij}u_3^{(k)} \\ &= \tfrac{1}{2}\left[(u_{i,j} + u_{j,i}) - (w_{,ij} + w_{,ji})z - (\chi_{p,j}\psi_{jp} + \chi_{p,i}\psi_{jp})\right] + k_{ij}w \qquad i,j = 1,2;\ p = 1,2,\ldots,8 \end{aligned} \tag{55}$$

The transverse shear strains are given by

$$2e_{i3}^{(k)} = u_{i,3}^{(k)} + u_{3,i}^{(k)} = u_{i,3}^{(k)} + w_{,i} = -(w_{,i} - \chi_p\psi_{ip}^{(k)}) + w_{,i} = \chi_p\psi_{ip}^{(k)} \qquad i = 1,2;\ p = 1,2,\ldots,8 \tag{56}$$

The strain due to the normal compression is equal to zero as hypothesis

$$e_{33}^{(k)} = 0 \tag{57}$$

The components of the stress tensor can be determined by substituting the strains (55)–(57) into the Hooke's law (11) as

$$\begin{aligned} \sigma_{11}^{(k)} &= A_{1i}^{(k)}e_{ir}^{(k)} + 2A_{16}^{(k)}e_{12}^{(k)} \\ &= A_{1i}^{(k)}(u_{i,r} - w_{,ir}z - \chi_{p,i}\psi_{rp}^{(k)} + k_{ir}w) + A_{16}^{(k)}\left[(u_{1,2} + u_{2,1}) - 2w_{,12}z - (\chi_{p,2}\psi_{1p}^{(k)} + \chi_{p,1}\psi_{2p}^{(k)}) + 2k_{12}w\right] \\ \sigma_{22}^{(k)} &\leftrightharpoons \sigma_{11}^{(k)} \\ \sigma_{12}^{(k)} &= A_{i6}^{(k)}e_{ir}^{(k)} + 2A_{66}^{(k)}e_{12}^{(k)} \\ &= A_{i6}^{(k)}(u_{i,r} - w_{,ir}z - \chi_{p,i}\psi_{rp}^{(k)} + k_{ir}w) + A_{66}^{(k)}\left[(u_{1,2} + u_{2,1}) - 2w_{,12}z - (\chi_{p,2}\psi_{1p}^{(k)} + \chi_{p,1}\psi_{2p}^{(k)}) + 2k_{12}w\right] \\ & i = 1,2;\ r = i;\ p = 1,2,\ldots,8 \end{aligned} \tag{58}$$

$$\begin{aligned} \sigma_{13}^{(k)} &= \chi_p(A_{55}^{(k)}\psi_{1p}^{(k)} + A_{54}^{(k)}\psi_{2p}^{(k)}) \\ \sigma_{23}^{(k)} &= \chi_p(A_{45}^{(k)}\psi_{1p}^{(k)} + A_{44}^{(k)}\psi_{2p}^{(k)}) \\ & p = 1,2,\ldots,8 \end{aligned} \tag{59}$$

$$\begin{aligned}\sigma_{33}^{(k)} &= A_{3i}^{(k)} e_{ir} + 2A_{66}^{(k)} e_{12}^{(k)} \\ &= A_{3i}^{(k)}(u_{i,r} - w_{,ir}z - \chi_{p,i}\psi_{rp}^{(k)} + k_{ir}w) + A_{66}^{(k)}\Big[(u_{1,2} + u_{2,1}) - w_{,12}z - (\chi_{p,2}\psi_{1p}^{(k)} + \chi_{p,1}\psi_{2p}^{(k)}) + 2k_{12}w\Big] \\ & i = 1,2; \;\; r = i; \;\; p = 1,2,\ldots,8\end{aligned} \tag{60}$$

It should be noted, that in accordance with the hypotheses (46) the transverse normal stresses $\sigma_{33}^{(k)}$ equal to zero. The Hooke's law can be used to find these stresses in the form of Eq. (60). However, they do not satisfy the equilibrium Eq. (14) and the conditions (2)–(4). The expressions (60) represent only that part of these stresses, which results from the tangential components $e_{ij}^{(k)}$ of the strain tensor. These stresses actually arise due to the Poisson's effect. The stresses $\sigma_{33}^{(k)}$, which satisfy the given conditions and the equilibrium equations, can be obtained from the expression (16). It is necessary by first to find from Eq. (21) the refined, comparing to the expressions (58), transverse shear stresses $\sigma_{i3}^{(k)}$ taking the expressions (2) for the tangential stresses $\sigma_{ij}^{(k)}$ into account. The refined transverse shear and normal stresses derived in such a manner will now correspond to the results of the non-classical theory just as earlier derived stresses $\sigma_{i3}^{(k)}$ in the form (40) corresponded to the results of the classical theory.

In the subsequent study of the stress–strain state of the shell it is necessary to obtain the set of governing equations and boundary conditions expressed in terms of the unknown functions of the reference surface u_i, w, χ_p, $(i = 1,2;\; p = 1,2,3,4)$. As this takes place, the functions of normal z are assumed as known and they are distribution functions of the stress–strain state components through the thickness of the multilayered anisotropic shell. These distribution functions are defined in a form which facilitates the satisfaction of the conditions on the external surfaces and the continuity conditions in between the layers when the reference surface is positioned arbitrarily through the thickness of the shell. Clearly, the governing equations are independent of the thicknesses, stiffnesses and other properties of the layers.

An important feature of the proposed non-classical model of the stress–strain state is the relation of its expressions with physical and mechanical characteristics of the anisotropic shell layers.

It should be noted that the association of the relations of the classical model (theory) with the physical and mechanical characteristics of the anisotropic material is solely by the stiffness parameters of the Hooke's law which ties together the stresses with the transverse shear strain. As this takes place, the displacements and the associated deformations are purely geometric relations.

In the proposed non-classical model the displacements (51), (54) and deformations (55) contain, in addition to the classical model, terms which take into account the influence of the transverse shear as well as their distribution functions through the thickness. These functions depend on both the stiffness characteristics and the shear compliance in the orthogonal directions.

This is due to the relationship between the physical and mechanical characteristics in the expressions for the transverse shear stresses (40) and transverse shear strain (44). These relations form the basis of the derivation of the shear deformations and displacements. Next, this association spreads to the normal stresses (58) and other relations of the non-classical theory. The availability of this relationship makes the proposed model different from the non-classical models in which the hypotheses of transverse shear stresses and their distribution functions are based on purely geometrical considerations, e.g. (Ambartsumyan, 1974), the models and theories of the first order shear deformation theory – FSDT.

5. Variational equation, equations of equilibrium and boundary conditions

5.1. Variational equation

The equations of equilibrium and the boundary conditions may be determined using the Lagrange's variational principle

$$\delta U - \delta H = 0 \tag{61}$$

where δU is the variation of the potential energy of the deformation and δH is the variation of the work done by the external forces.

For a laminated shell we consider the tangential and normal components of the stress and strain tensors and then the variation of the potential energy may be given by

$$\delta U = \iiint_V \left[\sigma_{ij}^{(k)}\delta e_{ij}^{(k)} + 2\sigma_{i3}^{(k)}\delta e_{i3}^{(k)} + \sigma_{33}^{(k)}\delta e_{33}^{(k)}\right]\mathrm{d}V, \quad i,j = 1,2 \tag{62}$$

Substituting strains from Eqs. (55)–(57) into Eq. (62), we can express the variation of the potential energy in terms of the displacements given by Eqs. (47) and (51), as

$$\delta U = \iint_S \left\{ \int_{a_0}^{a_n} \left[\sigma_{ij}^{(k)}(\delta u_{i,j} - z\delta w_{,ij} - \psi_{ip}^{(k)}\delta\chi_{p,j} + k_{ij}\delta w) + \sigma_{i3}^{(k)}(\psi_{ip}^{(k)}\delta\chi_p)\right]\mathrm{d}z\right\}\mathrm{d}S,$$
$$i,j = 1,2; \quad p = 1,2,3,4 \tag{63}$$

where S is the two-dimensional domain of the shell surface. It is noted that the variations of the known functions which have subscripts $p = 5, 6, 7, 8$ are equal to zero. In the following derivations we will replace index p with index f.

Using a notation similar to that of the classical theory we may now consider the following integral characteristics of stresses

$$\begin{aligned} &N_{ij} = \int_{a_0}^{a_n} \sigma_{ij}^{(k)}\,\mathrm{d}z; \qquad M_{ij} = \int_{a_0}^{a_n} \sigma_{ij}^{(k)} z\,\mathrm{d}z \\ &N_{ij}^{(rf)} = \int_{a_0}^{a_n} \sigma_{ij}\psi_{rf}^{(k)}\,\mathrm{d}z; \qquad Q_i^{(rf)} = \int_{a_0}^{a_n} \sigma_{i3}^{(k)}\psi_{rf}^{(k)}\,\mathrm{d}z \\ &i,j = 1,2; \quad r = i; \quad f = p = 1,2,3,4 \end{aligned} \tag{64}$$

Substituting (64) into Eq. (63), and using Ostrogradsky–Gauss theorem, we obtain the expression for the variation of the potential energy as

$$\begin{aligned} \delta U = &-\iint_S \left[N_{ij,j}\delta u_i + (M_{ij,ij} - k_{ij}N_{ij})\delta w - (N_{ij,j}^{(if)} + Q_i^{(if)})\delta\chi_p\right]\mathrm{d}S + \int_L \Big[(N_{hh}\delta u_h + N_{hl}\delta u_l) \\ &+ (M_{hh,h} + 2M_{hl,l})\delta w - M_{hh}\delta w_{,h} - N_{hh}^{(hf)}\delta\chi_f^{(h)} - N_{hl}^{(lf)}\delta\chi_f^{(l)}\Big]\mathrm{d}L + [M_{hl}\delta w]_{L_1}^{L_2} \end{aligned} \tag{65}$$

where h and l are normal and tangent to the boundary L of the domain of the shell, respectively. For the forces on the boundary of the domain it was assumed that h and l are equivalent to i and j in Eq. (64).

The variation of the work of the external loading we obtain as

$$\begin{aligned} \delta H &= \iint_S (p_i^-\delta u_i^{(1)} + p_i^+\delta u_i^{(n)} + p_3^-\delta u_3^{(1)} + p_3^+\delta u_3^{(n)})\,\mathrm{d}S \\ &= \iint_S \left\{p_i^-[\delta u_i - a_0\delta w_{,i} - \psi_{if}^{(1)}(a_0)\delta\chi_f] + p_i^+[\delta u_i - a_n\delta w_{,i} - \psi_{if}^{(n)}(a_n)\delta\chi_f] + (p_3^- + p_3^+)\delta w\right\}\mathrm{d}S \\ &= \iint_S \left\{(p_i^- + p_i^+)\delta u_i + [(p_{i,i}^- a_0 + p_{i,i}^+ a_n) + p_3]\delta w - [p_i^-\psi_{if}^{(1)}(a_0) + p_i^+\psi_{if}^{(n)}(a_n)]\delta\chi_f\right\}\mathrm{d}S \\ &\quad - \int_L [(p_h^- a_0 + p_h^+ a_n)\delta w]\,\mathrm{d}L \qquad i = 1,2; \quad f = 1,2,3,4 \end{aligned} \tag{66}$$

Substituting the variations (65) and (66) into Eq. (61), we derive the following variational equation

$$\int\int_S \Big\{[N_{ij,j} + (p_i^- + p_i^+)]\delta u_i + [M_{ij,ij} - k_{ij}N_{ij} + (p_{i,i}^- a_0 + p_{i,i}^+ a_n + p_3)]\delta w - [N_{ij,j}^{(if)} + Q_i^{(if)} + p_i^- \psi_{if}^{(1)}(a_0)$$
$$+ p_i^+ \psi_{if}^{(n)}(a_n)]\chi_f\Big\}\,\mathrm{d}S - \int_L \Big\{(N_{hh}\delta u_h + N_{hl}\delta u_l) + [(M_{hh,h} + 2M_{hl,l}) + (p_h^- a_0 + p_h^+ a_n)]\delta w - M_{hh}\delta w_{,h}$$
$$- N_{hh}^{(hf)}\delta\chi_f^{(h)} - N_{hl}^{(lf)}\delta\chi_f^{(l)}\Big\}\,\mathrm{d}L - [M_{hl}\delta w]_{L_1}^{L_2} = 0 \qquad i,j = 1,2;\ \ f = 1,2,3,4 \tag{67}$$

5.2. Equations of equilibrium and boundary conditions

The variations of independent functions u_i, w, χ_p which determine the displacements in the shell, have arbitrary values everywhere over the domain of the shell excluding the boundary and, consequently, they cannot be equal to zero. Equating the multipliers of the variations in the first integral of Eq. (67) to zero, we obtain the system of equations of equilibrium of the shell as

$$\begin{aligned}
&N_{ij,j} + (p_i^- + p_i^+) = 0\\
&M_{ij,ij} - k_{ij}N_{ij} + (p_{i,i}^- a_0 + p_{i,i}^+ a_n + p_3) = 0\\
&N_{ij,j}^{(if)} + Q_i^{(if)} + [p_i^- \psi_{if}^{(1)}(a_0) + p_i^+ \psi_{if}^{(n)}(a_n)] = 0\\
&\quad i,j = 1,2;\ \ f = 1,2,3,4
\end{aligned} \tag{68}$$

The boundary conditions follow from the boundary integral in the equation (67) and they may be written as follows

- kinematic conditions

$$\begin{aligned}
u_h &= 0; \quad u_l = 0\\
w &= 0; \quad w_{,h} = 0\\
\chi_f^{(h)} &= 0; \quad \chi_f^{(l)} = 0
\end{aligned} \tag{69}$$

- corresponded static conditions

$$\begin{aligned}
&N_{hh} = 0; \quad N_{lh} = 0\\
&M_{hh,h} + 2M_{hl,l} + (p_h^- a_0 + p_h^+ a_n) = 0; \qquad M_{hh} = 0\\
&N_{hh}^{(hf)} = 0; \quad N_{hl}^{(lf)} = 0
\end{aligned} \tag{70}$$

where $f = 1, 2$ if $h, l = 1$; $f = 3, 4$ if $h, l = 2$.

There are eight boundary conditions, which is the same as the order of the system of equations (68). A detailed interpretation of the boundary conditions may be given as in Piskunov et al. (1987, 1993) where a simple case of isotropic laminated plates and shells is considered.

It is clear that the first and second expressions in Eq. (68) both are the equations of the classical theory (26). The rest of the equations take into account the shear strain effect. Eq. (68) constitute the system of equilibrium equations of the non-classical higher-order (third order) theory of the anisotropic laminated shallow shells. And Eqs. (69) and (70) are the boundary conditions for this system.

6. Generalized forces and moments and system of governing equations

6.1. Forces and moments

Let us rewrite the integral characteristics of stresses given by Eq. (64) and use the expressions for the stresses for the kth layer given by Eqs. (58) and (59). Then we have for the tangential forces

$$
\begin{aligned}
N_{11} &= B_{1i}(u_{i,r} + k_{ir}w) + B_{16}(u_{1,2} + u_{2,1} + 2k_{12}w) - C_{1i}w_{,ir} - 2C_{16}w_{,12} - (B_{11}^{(1p)} + B_{16}^{(2p)})\chi_{p,1} \\
&\quad - (B_{12}^{(2p)} + B_{16}^{(1p)})\chi_{p,2} \\
N_{22} &\leftrightharpoons N_{11} \\
N_{12} &= B_{i6}(u_{i,r} + k_{ir}w) + B_{66}(u_{1,2} + u_{2,1} + 2k_{12}w) - C_{i6}w_{,ir} - 2C_{66}w_{,12} - (B_{16}^{(1p)} + B_{66}^{(2p)})\chi_{p,1} \\
&\quad - (B_{26}^{(2p)} + B_{66}^{(1p)})\chi_{p,2} \\
N_{21} &\leftrightharpoons N_{12} \qquad i = 1, 2; \;\; r = i
\end{aligned}
\tag{71}
$$

for the moments

$$
\begin{aligned}
M_{11} &= C_{1i}(u_{i,r} + k_{ir}w) + C_{16}(u_{1,2} + u_{2,1} + 2k_{12}w) - D_{1i}w_{,ir} - 2D_{16}w_{,12} - (C_{11}^{(1p)} + C_{16}^{(2p)})\chi_{p,1} \\
&\quad - (C_{12}^{(2p)} + C_{16}^{(1p)})\chi_{p,2} \\
M_{22} &\leftrightharpoons M_{11} \\
M_{12} &= C_{i6}(u_{i,r} + k_{ir}w) + C_{66}(u_{1,2} + u_{2,1} + 2k_{12}w) - D_{i6}w_{,ir} - 2D_{66}w_{,12} - (C_{16}^{(1p)} + C_{66}^{(2p)})\chi_{p,1} \\
&\quad - (C_{26}^{(2p)} + C_{66}^{(1p)})\chi_{p,2} \\
M_{21} &\leftrightharpoons M_{12} \qquad i = 1, 2; \;\; r = i
\end{aligned}
\tag{72}
$$

for the higher-order forces

$$
\begin{aligned}
N_{11}^{(1f)} &= B_{1i}^{(1f)}(u_{i,r} + k_{ir}w) + B_{16}^{(1f)}(u_{1,2} + u_{2,1} + 2k_{12}w) - C_{1i}^{(1f)}w_{,ir} - 2C_{16}^{(1f)}w_{,12} - (D_{11}^{(1f1p)} + D_{16}^{(1f2p)})\chi_{p,1} \\
&\quad - (D_{12}^{(1f2p)} + D_{16}^{(1f1p)})\chi_{p,2} \\
N_{22}^{(2f)} &\leftrightharpoons N_{11}^{(1f)} \\
N_{12}^{(1f)} &= B_{i6}^{(1f)}(u_{i,r} + k_{ir}w) + B_{66}^{(1f)}(u_{1,2} + u_{2,1} + 2k_{12}w) - C_{i6}^{(1f)}w_{,ir} - 2C_{66}^{(1f)}w_{,12} - (D_{16}^{(1f1p)} + D_{66}^{(1f2p)})\chi_{p,1} \\
&\quad - (D_{26}^{(1f2p)} + D_{66}^{(1f1p)})\chi_{p,2} \\
N_{21}^{(2f)} &\leftrightharpoons N_{12}^{(1f)} \qquad i = 1, 2; \;\; r = i
\end{aligned}
\tag{73}
$$

and for the shear forces

$$
\begin{aligned}
Q_1^{(1f)} &= \chi_p(R_{55}^{(1f1p)} + R_{54}^{(1f2p)}) \\
Q_2^{(2f)} &= \chi_p(R_{45}^{(2f1p)} + R_{44}^{(2f2p)})
\end{aligned}
\tag{74}
$$

In Eqs. (71)–(74) we have $f = 1, 2, 3, 4$; $p = 1, 2, \ldots, 8$, and we also assume summation over i and p.

The equations for the forces and moments include the integrated stiffnesses of the laminated shell given by

$$
\begin{aligned}
&B_{qs} = \int_{a_0}^{a_n} A_{qs}^{(k)}\,\mathrm{d}z; \qquad C_{qs} = \int_{a_0}^{a_n} A_{qs}^{(k)} z\,\mathrm{d}z \\
&B_{qs}^{(tp)} = \int_{a_0}^{a_n} A_{qs}^{(k)}\psi_{tp}^{(k)}\,\mathrm{d}z; \qquad C_{qs}^{(tp)} = \int_{a_0}^{a_n} A_{qs}^{(k)} z\psi_{tp}^{(k)}\,\mathrm{d}z \\
&D_{qs} = \int_{a_0}^{a_n} A_{qs}^{(k)} z^2\,\mathrm{d}z; \qquad D_{qs}^{(rftp)} = \int_{a_0}^{a_n} A_{qs}^{(k)}\psi_{rf}^{(k)}\psi_{tp}^{(k)}\,\mathrm{d}z \\
&R_{mn}^{(rftp)} = \int_{a_0}^{a_n} A_{mn}^{(k)}\psi_{rf}^{(k)}\psi_{tp}^{(k)}\,\mathrm{d}z \\
&\quad q,s = 1,2,6;\;\; m,n = 4,5;\;\; r,t = 1,2;\;\; f = 1,2,3,4;\;\; p = 1,2,\ldots,8
\end{aligned}
\tag{75}
$$

6.2. System of governing differential equations

Substituting expressions for the forces and moments (71)–(74) into the system of equations (68) we obtain the system of governing differential equations expressed in terms of the unknown functions. The system may be written in the following form

$$[D]\{U\} = [D_X]\{X\} + [F]\{P\} \tag{76}$$

where $[D]$ is the matrix of differential operators over the vector of unknown functions of the reference surface, which is given by

$$\{U\} = \{u_i; w; \chi_p\}^{\mathrm{T}}, \quad i = 1,2;\;\; p = 1,2,3,4 \tag{77}$$

$[D_X]$ is the matrix of differential operators over the vector of known functions defined by (53) and given as

$$\{X\} = \{\chi_p\}^{\mathrm{T}}, \quad p = 5,6,7,8 \tag{78}$$

and $[F]$ is the matrix of differential operators over the vector of given loads which is

$$\{P\} = \{p_i^{\mp}; p_3\}^{\mathrm{T}} \tag{79}$$

Finally, the left part of the equation (76) may be written in the following form

$$[D]\{U\} = \begin{bmatrix} [D_{11}] & [D_{12}] & [D_{13}] & [D_{1p}] \\ [D_{21}] & [D_{22}] & [D_{23}] & [D_{2p}] \\ [D_{31}] & [D_{32}] & [D_{33}] & [D_{3p}] \\ [D_{f1}] & [D_{f2}] & [D_{f3}] & [D_{fp}] \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ w \\ \chi_p \end{Bmatrix} \tag{80}$$

In order to multiply the matrix by the vector it is necessary to expand it over the index $f = 1, 2, 3, 4$ and make summation over the index $p = 1, 2, 3, 4$

$$
\begin{aligned}
&[D_{11}] = B_{11}(\cdots)_{,11} + 2B_{16}(\cdots)_{,12} + B_{66}(\cdots)_{,22} \\
&[D_{12}] = [D_{21}] = B_{16}(\cdots)_{,11} + (B_{12} + B_{66})(\cdots)_{,12} + B_{26}(\cdots)_{,22} \\
&[D_{22}] = B_{22}(\cdots)_{,22} + 2B_{26}(\cdots)_{,12} + B_{66}(\cdots)_{,11}
\end{aligned}
$$

$$
\begin{aligned}
[D_{13}] &= [D_{31}] \\
&= -\Big\{[C_{11}(\cdots)_{,11} + (C_{12} + 2C_{66})(\cdots)_{,22} - (B_{11}k_{11} + B_{12}k_{22} + 2B_{16}k_{12})(\cdots)]_{,1} \\
&\quad + [3C_{16}(\cdots)_{,11} + C_{26}(\cdots)_{,22} - (B_{16}k_{11} + B_{26}k_{22} + 2B_{66}k_{12})(\cdots)]_{,2}\Big\}
\end{aligned}
$$

$$[D_{23}] = [D_{32}] = -\Big\{[C_{22}(\cdots)_{,22} + (C_{12} + 2C_{66})(\cdots)_{,11} - (B_{22}k_{22} + B_{12}k_{11} + 2B_{26}k_{12})(\cdots)]_{,2} + [3C_{26}(\cdots)_{,22} + C_{16}(\cdots)_{,11} - (B_{26}k_{22} + B_{16}k_{11} + 2B_{66}k_{12})(\cdots)]_{,1}\Big\}$$

$$[D_{33}] = [D_{11}(\cdots)_{,11} + D_{12}(\cdots)_{,22} + 2D_{16}(\cdots)_{,12}]_{,11} + [D_{22}(\cdots)_{,22} + D_{21}(\cdots)_{,11} + 2D_{26}(\cdots)_{,12}]_{,22} + 2[D_{16}(\cdots)_{,11} + D_{26}(\cdots)_{,22} + 2D_{66}(\cdots)_{,12}]_{,12} - 2[(k_{11}C_{11} + k_{22}C_{12} + 2k_{12}C_{16})(\cdots)_{,11} + (k_{22}C_{22} + k_{11}C_{21} + 2k_{12}C_{26})(\cdots)_{,22} + 2(k_{11}C_{16} + k_{22}C_{26} + 2k_{12}C_{66})(\cdots)_{,12} + k_{11}(k_{11}B_{11} + k_{22}B_{22} + 2k_{12}B_{16}) + k_{22}(k_{22}B_{22} + k_{11}B_{11} + 2k_{12}B_{26}) + 2k_{12}(k_{11}B_{16} + k_{22}B_{26} + 2k_{12}B_{66})(\cdots)]$$

$$[D_{1p}] = [D_{f1}] = -\Big[\Big(B_{11}^{(1p)} + B_{16}^{(2p)}\Big)(\cdots)_{,11} + \Big(B_{26}^{(2p)} + B_{66}^{(1p)}\Big)(\cdots)_{,22} + \Big(B_{12}^{(2p)} + 2B_{16}^{(1p)} + B_{66}^{(2p)}\Big)(\cdots)_{,12}\Big]$$

$$[D_{2p}] = [D_{f2}] = -\Big[\Big(B_{22}^{(2p)} + B_{26}^{(1p)}\Big)(\cdots)_{,22} + \Big(B_{16}^{(1p)} + B_{66}^{(2p)}\Big)(\cdots)_{,11} + \Big(B_{21}^{(1p)} + 2B_{26}^{(2p)} + B_{66}^{(1p)}\Big)(\cdots)_{,12}\Big]$$

$$[D_{3p}] = [D_{f3}] = \Big[\Big(C_{11}^{(1p)} + C_{16}^{(2p)}\Big)(\cdots)_{,1} + \Big(C_{12}^{(2p)} + C_{16}^{(1p)}\Big)(\cdots)_{,2}\Big]_{,11} + \Big[\Big(C_{22}^{(2p)} + C_{26}^{(1p)}\Big)(\cdots)_{,2} + \Big(C_{12}^{(1p)} + C_{26}^{(2p)}\Big)(\cdots)_{,1}\Big]_{,22} + 2\Big[\Big(C_{16}^{(1p)} + C_{66}^{(2p)}\Big)(\cdots)_{,1} + \Big(C_{26}^{(2p)} + C_{66}^{(1p)}\Big)(\cdots)_{,2}\Big]_{,12} - \Big[k_{11}\Big(B_{11}^{(1p)} + B_{16}^{(2p)}\Big) + k_{22}\Big(B_{12}^{(1p)} + B_{26}^{(2p)}\Big) + 2k_{12}\Big(B_{16}^{(1p)} + B_{66}^{(2p)}\Big)\Big](\cdots)_{,1} - \Big[k_{11}\Big(B_{12}^{(2p)} + B_{16}^{(1p)}\Big) + k_{22}\Big(B_{22}^{(2p)} + B_{26}^{(1p)}\Big) + 2k_{12}\Big(B_{26}^{(2p)} + B_{66}^{(1p)}\Big)\Big](\cdots)_{,2}$$

$$[D_{fp}] = \Big[\Big(D_{11}^{(1f1p)} + D_{16}^{(1f2p)}\Big) + \Big(D_{16}^{(2f1p)} + D_{66}^{(2f2p)}\Big)\Big](\cdots)_{,11} + \Big[\Big(D_{22}^{(2f2p)} + D_{26}^{(2f1p)}\Big) + \Big(D_{26}^{(1f2p)} + D_{66}^{(1f1p)}\Big)\Big](\cdots)_{,22} + \Big[\Big(D_{12}^{(1f2p)} + 2D_{16}^{(1f1p)} + D_{66}^{(1f2p)}\Big) + \Big(D_{12}^{(2f1p)} + 2D_{26}^{(2f2p)} + D_{66}^{(2f1p)}\Big)\Big](\cdots)_{,12} - \Big(R_{44}^{(2f2p)} + R_{45}^{(2f1p)} + R_{54}^{(1f2p)} + R_{55}^{(1f1p)}\Big)(\cdots),$$
$$f, p = 1, 2, 3, 4 \tag{81}$$

It is noted that submatrices $[D_{f1}]$ and $[D_{f2}]$ can be obtained from submatrices $[D_{1p}]$ and $[D_{2p}]$ by replacing index p with f.

The right-hand side of Eq. (76) is formed by two terms. The first item has the following form

$$[D_X]\{X\} = -\begin{bmatrix}[D_{1p}]\\ [D_{2p}]\\ [D_{3p}]\\ [D_{fp}]\end{bmatrix}\left\{\chi_p = \begin{vmatrix}p_1^-\\ p_1^+\\ p_2^-\\ p_2^+\end{vmatrix}\right\} \quad f = 1, 2, 3, 4;\ \ p = 5, 6, 7, 8 \tag{82}$$

where the submatrices $[D_{1p}]\ldots[D_{fp}]$ are defined in the expressions (81).

The second item is given as follows

$$[F]\{P\} = \begin{bmatrix} -(\cdots) & -(\cdots) & - & - & - \\ - & - & -(\cdots) & -(\cdots) & - \\ a_0(\cdots)_{,1} & a_n(\cdots)_{,1} & a_0(\cdots)_{,2} & a_n(\cdots)_{,2} & (\cdots) \\ \psi_{1f}^{(1)}(a_0)(\cdots) & \psi_{1f}^{(n)}(a_n)(\cdots) & \psi_{2f}^{(1)}(a_0)(\cdots) & \psi_{2f}^{(n)}(a_n)(\cdots) & - \end{bmatrix} \begin{Bmatrix} p_1^- \\ p_1^+ \\ p_2^- \\ p_2^+ \\ p_3 \end{Bmatrix} \tag{83}$$

$f = 1, 2, 3, 4$

The total number of equations in the system (76) is equal to seven. The total order of the equations is equal to 16 and corresponds to the number of boundary conditions expressed by Eq. (69) and are equal to eight.

7. Analytical solution and special cases

7.1. General case

Let us investigate a possibility of obtaining an analytical solution of the system of differential equations (76). It is apparent that this solution is only possible for the simplest cases. We will consider the case of a supported shell with a rectangular plan view. The Navier approach can be used in this case by expanding the applied loads and unknown functions in double Fourier trigonometric series. For simplicity, from here on, we will introduce the following notations for the series:

$$\begin{aligned} &\sin\frac{m\pi}{a_1}x_1 = \sin\lambda_m x_1 = S_m; \qquad \cos\frac{m\pi}{a_1}x_1 = \cos\lambda_m x_1 = C_m \\ &\sin\frac{n\pi}{a_2}x_2 = \sin\gamma_n x_2 = S_n; \qquad \cos\frac{n^{(k)}}{a_2}x_2 = \cos\gamma_n x_2 = C_n \end{aligned} \tag{84}$$

We consider two cases of the shell support, for which Fourier series are formed variously depending upon the boundary conditions.

The first case is a hinged movable (free) support, when the edges of the shell can freely move along the normal to the edges. Tangential displacements along the edges are not permitted. The boundary conditions, accordingly to Eqs. (69) and (70), are the following

$$\begin{aligned} &\text{for } x_1 = 0; \quad a_1 : u_2 = 0;\ N_{11} = 0;\ w = 0;\ M_{11} = 0;\ \chi_3^{(2)} = \chi_4^{(2)} = 0;\ N_{11}^{(11)} = N_{11}^{(21)} = 0 \\ &\text{for } x_2 = 0; \quad a_2 : u_1 = 0;\ N_{22} = 0;\ w = 0;\ M_{22} = 0;\ \chi_1^{(1)} = \chi_2^{(2)} = 0;\ N_{22}^{(23)} = N_{22}^{(24)} = 0 \end{aligned} \tag{85}$$

The second lines in the conditions (85) model end diaphragms on the corresponded edges, which prevent the transverse shears in the planes of the ends and allow the shear along the normal to the edges. Various variants of such diaphragms and their modeling using the boundary conditions may be given as in Piskunov et al. (1987, 1993) for the isotropic shells.

In order to satisfy the conditions (85) the following expansions in the series are taken

$$\begin{aligned} &u_1 = \sum_m\sum_n U_{1mn}C_mS_n; \qquad u_2 = \sum_m\sum_n U_{2mn}S_mC_n; \qquad w = \sum_m\sum_n W_{mn}S_mS_n \\ &[\chi_1;\ \chi_2] = \sum_m\sum_n [X_{1mn};\ X_{2mn}]C_mS_n \\ &[\chi_3;\ \chi_4] = \sum_m\sum_n [X_{3mn};\ X_{4mn}]S_mC_n \end{aligned} \tag{86}$$

Correspondingly, expansions for the loads are as follows

$$p_1^{\mp} = \sum_m \sum_n P_{1mn}^{\mp} C_m S_n; \qquad p_2^{\mp} = \sum_m \sum_n P_{2mn}^{\mp} S_m C_n; \qquad p_3 = \sum_m \sum_n P_{3mn} S_m S_n \tag{87}$$

The second case is a hinged support when the shell edges are not free to move in the normal directions to its contour. Tangential displacements along the edges are possible. This is the case of the hinged immovable support of the shell. Here the boundary conditions, according to Eqs. (69) and (70) are as follows

$$\begin{aligned} &\text{for } x_1 = 0; \quad a_1 : u_1 = 0;\; N_{12} = 0;\; w = 0;\; M_{11} = 0;\; \chi_3^{(2)} = \chi_4^{(2)} = 0;\; N_{11}^{(11)} = N_{11}^{(21)} = 0 \\ &\text{for } x_2 = 0; \quad a_2 : u_2 = 0;\; N_{21} = 0;\; w = 0;\; M_{22} = 0;\; \chi_1^{(1)} = \chi_2^{(1)} = 0;\; N_{22}^{(23)} = N_{22}^{(24)} = 0 \end{aligned} \tag{88}$$

In satisfying the conditions (88) we will use the following expressions of the unknown functions in Fourier series

$$\begin{aligned} &u_1 = \sum_m \sum_n U_{1mn} S_m C_n; \qquad u_2 = \sum_m \sum_n U_{2mn} C_m S_n; \qquad w = \sum_m \sum_n W_{mn} S_m S_n \\ &[\chi_1;\; \chi_2] = \sum_m \sum_n [X_{1mn};\; X_{2mn}] C_m S_n \\ &[\chi_3;\; \chi_4] = \sum_m \sum_n [X_{3mn};\; X_{4mn}] S_m C_n \end{aligned} \tag{89}$$

Corresponded expansions for the loads are as follows

$$p_1^{\mp} = \sum_m \sum_n P_{1mn}^{\mp} S_m C_n; \qquad p_2^{\mp} = \sum_m \sum_n P_{2mn}^{\mp} C_m S_n; \qquad p_3 = \sum_m \sum_n P_{3mn} S_m S_n \tag{90}$$

The expressions (86) and (87), (89) and (90) contain amplitudes of sought functions U_{imn}, W_{mn}, X_{pmn} and the load functions $P_{imn}^{\pm}$, $P_{3mn}^{\pm}$.

Expansions of the sought functions and the functions of loads for each case of the boundary conditions can be substituted into the expressions for submatrices (81) of the system of differential equations (76). The object of such substitution is to obtain the system of algebraic equations for the amplitudes of sought functions when the load amplitudes are known. In doing so, the trigonometric multipliers for each forming equation must be retained. However, one can make sure that none of considered variants of expansions does not allow to obtain a system of equations which could be free from the trigonometric multipliers. The trigonometric multipliers cannot be canceled and the system of algebraic equation cannot be formed. Consequently, an analytical solution in Fourier series for the considered equations is impossible. Therefore, it is impossible to obtain the analytical solution for an anisotropic shell of arbitrary layered structure through the thickness even within the framework of the classical theory. This fact is known, for example, from Savoia and Reddy (1992), Noor and Burton (1990b).

The problem of canceling terms of one or another of generalized stiffness characteristics and obtaining of the system of equations, which could be solved, must be considered in connection with the structure of laminated shell, and also, with geometry of its surface. Next we consider special cases of the structure, for which the solution is possible.

7.2. Analysis of special cases: cross-ply and angle-ply laminates

There are two significant special cases of the structure of laminated plate or shell in practice. The first case is cross-ply laminated structure (see Fig. 2a) formed with layers reinforced at angles 0° and 90° to the orthogonal axes of coordinate x_1 and x_2. The layers of shell with such a structure are orthotropic. The stiffness parameters $A_{16}^{(k)}$, $A_{26}^{(k)}$, $A_{36}^{(k)}$, $A_{45}^{(k)}$, $A_{54}^{(k)}$ for such layers are equal to zero in the Hooke's law (11).

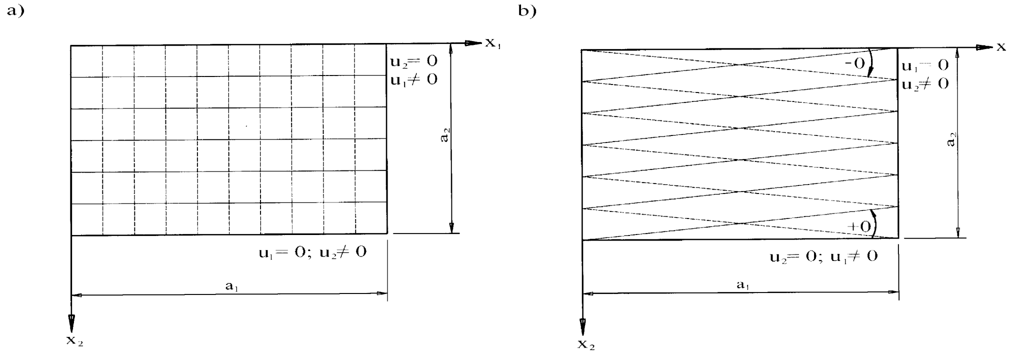

Fig. 2. Cross-ply (a) and angle-ply (b) laminated structures and corresponding boundary conditions.

Consequently, the distribution functions of the stiffness characteristics set in the formulae (34) and (35) will be also reduced to zero.

$$f_{16}^{(k)} = f_{26}^{(k)} = F_{16}^{(k)} = F_{26}^{(k)} = 0; \qquad B_{16} = B_{26} = C_{16} = C_{26} = 0 \tag{91}$$

In the case when the shell has a symmetrical structure about the mean surface, then additionally we have

$$C_{11} = C_{12} = C_{22} = C_{66} = 0 \tag{92}$$

The second case is angle-ply laminated structure (see Fig. 2b) formed with layers reinforced at the same angles but opposite in sign $(-\theta, +\theta)$ to the coordinate axes x_1 and x_2. The layers of such a shell are anisotropic and all the stiffness parameters remain in the Hooke's law (11).

In this case, of interest is antisymmetric structure through the thickness about the mean surface, that is, each layer with fiber orientation $(-\theta)$ has corresponded layer with respect to the main surface with fiber orientation $(+\theta)$. For this structure the stiffness characteristics $A_{16}^{(k)}, A_{26}^{(k)}, A_{36}^{(k)}, A_{45}^{(k)}, A_{54}^{(k)}$ of the given layers are opposite in sign (antisymmetric), then from the formula (35) follows that $B_{16} = B_{26} = 0$, $C_{16} \neq 0$, $C_{26} \neq 0$. Since the stiffness characteristics of the $A_{21}^{(k)}$, $A_{12}^{(k)}, \ldots$ type for the same layers have the same sign (symmetrical), then the relations (92) are also true.

Attention must be given that in both special cases we have the same "zero" characteristics for $B_{16} = B_{26} = 0$. We will use this property in the relations (35). First we consider cross-ply laminates. The stiffness characteristics B_{16} and B_{26} are present in the third and fourth relations. As we also have $C_{16} = C_{26} = 0$, then these relations are satisfied by equality $(0 \equiv 0)$.

For angle-ply laminates $C_{16} \neq 0$ and $C_{26} \neq 0$. It means, that the considered relations (35) will be identically satisfied if we have for the stresses $\sigma_{13}^{(k)}$ that $\kappa_{11,2} = \kappa_{22,2} = 0$. Similarly, in the expressions for stresses $\sigma_{23}^{(k)}$ $\kappa_{22,1} = \kappa_{11,1} = 0$. Taking into account these relations in the expressions (40) for the transverse shear stresses we obtain

$$\begin{aligned} \sigma_{13}^{(k)} &= p_1^- \varphi_{1k}^- + p_1^+ \varphi_{1k}^+ - \left(\kappa_{11,1}\varphi_{11}^{(k)} + \kappa_{22,1}\varphi_{12}^{(k)}\right) \\ \sigma_{23}^{(k)} &= p_2^- \varphi_{2k}^- + p_2^+ \varphi_{2k}^+ - \left(\kappa_{22,2}\varphi_{21}^{(k)} + \kappa_{11,2}\varphi_{22}^{(k)}\right) \end{aligned} \tag{93}$$

These expressions are also true for cross-ply laminates, since after substitution of Eq. (91) into Eq. (39) we obtain "zero" functions $\varphi_{13}^{(k)} = \varphi_{23}^{(k)} = \varphi_{14}^{(k)} = \varphi_{24}^{(k)} = 0$. Thus, the expressions for the transverse shear stresses for both considered special cases are the same. Compared to Eq. (40), the simplified equations (93) retain the relationship with the physical and mechanical characteristics of the anisotropic layers and, correspondingly, transfer this relationship to the next relations, first of all to the transverse shear deformations.

Let us now consider expressions (44) for the transverse shear deformations. In case of angle-ply laminates they fully hold their form.

For cross-ply laminates we take into account the fact that $a_{45}^{(k)} = a_{54}^{(k)} = 0$. Therefore, we obtain from formulae (42) and (43)

$$\begin{aligned} \Psi_{13}^{(k)} &= \Psi_{14}^{(k)} = \Psi_{21}^{(k)} = \Psi_{22}^{(k)} = 0 \\ \Psi_{17}^{(k)} &= \Psi_{18}^{(k)} = \Psi_{25}^{(k)} = \Psi_{26}^{(k)} = 0 \end{aligned} \tag{94}$$

In accordance with formulae (94) and the expressions (45) for the transverse shear deformations are simplified

$$\begin{aligned} 2e_{13}^{(k)} &= \kappa_{11,1}\Psi_{11}^{(k)} + \kappa_{22,1}\Psi_{12}^{(k)} + p_1^-\Psi_{15}^{(k)} + p_1^+\Psi_{16}^{(k)} \\ 2e_{23}^{(k)} &= \kappa_{11,2}\Psi_{23}^{(k)} + \kappa_{22,2}\Psi_{24}^{(k)} + p_2^-\Psi_{27}^{(k)} + p_2^+\Psi_{28}^{(k)} \end{aligned} \tag{95}$$

The expressions (51) for the tangential displacements will contain, as in the general case, the distribution functions through the thickness $\psi_{ip}^{(k)}$ which are determined accordingly to the relations (50) through the distribution functions of deformations. We have in the case of cross-ply laminates

$$\psi_{13}^{(k)} = \psi_{14}^{(k)} = \psi_{21}^{(k)} = \psi_{22}^{(k)} = 0; \qquad \psi_{17}^{(k)} = \psi_{18}^{(k)} = \psi_{25}^{(k)} = \psi_{26}^{(k)} = 0 \tag{96}$$

For angle-ply laminates all the functions $\psi_{ip}^{(k)}$ stay the same. It is noted that for both special cases we will have the following distribution functions

$$\psi_{11}^{(k)}, \psi_{12}^{(k)}, \psi_{23}^{(k)}, \psi_{24}^{(k)}, \psi_{15}^{(k)}, \psi_{16}^{(k)}, \psi_{27}^{(k)}, \psi_{28}^{(k)}$$

Only in the case of angle-ply laminates the following even functions exist

$$\psi_{13}^{(k)}, \psi_{14}^{(k)}, \psi_{21}^{(k)}, \psi_{22}^{(k)}$$

and odd functions

$$\psi_{17}^{(k)}, \psi_{18}^{(k)}, \psi_{25}^{(k)}, \psi_{26}^{(k)}$$

In both cases the general expression for tangential displacements (51) and the functions of coordinate surface set with the relations (52) and (53) stay the same. The expression for the tangential displacements (54) of the non-classical theory is also unchanged. In this expression and in all other relations of the non-classical theory the "zero" distribution functions must be taken account according to (96). Property of the distribution functions $\psi_{ip}^{(k)}$, their eveness or oddness allow to find out which of generalized stiffness characteristics, determined by formulae (75), for each considered laminates are "zero" and which are remained.

In Table 1 there are given those stiffness characteristics which are remained in the submatrices (81) of the differential equations (76) for special cases: cross-ply and angle-ply laminates. The stiffness without upper indices belong to both classical and non-classical theories and determine resistance of the laminated shell when the transverse shear takes place.

The stiffnesses with upper indices $p = 1, 2, 3, 4$ therewith are constituents of submatrices of the left-hand side of the system of differential governing equations (76). The stiffnesses with indices $p = 5, 6, 7, 8$ are constituents of submatrices of the right-hand side of this system.

By this means using the general form of the system of differential equations (76), its separate parts (80), (82), (83), expressions for submatrices (81) and Table 1 we can compile the system of differential equations for the special cases: cross-ply and angle-ply laminates.

Table 1
Stiffness charecteristics of the laminated shell for the special cases

	Cross-ply laminates		Angle-ply laminates	
Classic and non-classic theories	$B_{11}, B_{22}, B_{12}, B_{66}$ $C_{11}, C_{22}, C_{12}, C_{66}$ $D_{11}, D_{22}, D_{12}, D_{66}$		$B_{11}, B_{22}, B_{12}, B_{66}$ C_{16}, C_{26} $D_{11}, D_{22}, D_{12}, D_{66}$	
Non-classic theory	$p = 1, 2, 5, 6$	$p = 3, 4, 7, 8$	$p = 1, 2, 5, 6$	$p = 3, 4, 7, 8$
$f \rightleftharpoons p$ = 1,2; 3,4	$B_{11}^{(1p)}, B_{12}^{(1p)}, B_{66}^{(1p)}$ $C_{11}^{(1p)}, C_{12}^{(1p)}, C_{66}^{(1p)}$	$B_{22}^{(2p)}, B_{12}^{(2p)}, B_{66}^{(2p)}$ $C_{22}^{(2p)}, C_{12}^{(2p)}, C_{66}^{(2p)}$	$B_{22}^{(2p)}, B_{12}^{(2p)}, B_{66}^{(2p)}$ $B_{16}^{(1p)}, B_{26}^{(1p)}$ $C_{11}^{(1p)}, C_{12}^{(1p)}, C_{66}^{(1p)}$ $C_{16}^{(2p)}, C_{26}^{(2p)}$	$B_{11}^{(1p)}, B_{12}^{(1p)}, B_{66}^{(1p)}$ $B_{16}^{(2p)}, B_{26}^{(2p)}$ $C_{22}^{(2p)}, C_{12}^{(2p)}, C_{66}^{(2p)}$ $C_{16}^{(1p)}, C_{26}^{(1p)}$
f = 1,2	$D_{11}^{(1f1p)}, D_{66}^{(1f1p)}$ $R_{55}^{(1f1p)}$	$D_{12}^{(1f2p)}, D_{66}^{(1f2p)}$	$D_{11}^{(1f1p)}, D_{22}^{(2f2p)}, D_{66}^{(1f1p)}$ $D_{66}^{(2f2p)}, D_{16}^{(1f2p)}, D_{16}^{(2f1p)}$ $D_{26}^{(1f2p)}, D_{26}^{(2f1p)}$ $R_{44}^{(2f2p)}, R_{45}^{(2f1p)}$ $R_{54}^{(1f2p)}, R_{55}^{(1f1p)}$	$D_{12}^{(1f2p)}, D_{12}^{(2f1p)}, D_{66}^{(1f2p)}$ $D_{66}^{(2f1p)}, D_{16}^{(1f1p)}, D_{26}^{(2f2p)}$
f = 3,4	$D_{12}^{(1p2f)}, D_{66}^{(1p2f)}$	$D_{22}^{(2f2p)}, D_{66}^{(2f2p)}$ $R_{44}^{(2f2p)}$	$D_{12}^{(1p2f)}, D_{12}^{(2p1f)}, D_{66}^{(1p2f)}$ $D_{66}^{(2p1f)}, D_{16}^{(1p1f)}, D_{26}^{(2p2f)}$	$D_{11}^{(1f1p)}, D_{22}^{(2f2p)}, D_{66}^{(1f1p)}$ $D_{66}^{(2f2p)}, D_{16}^{(1f2p)}, D_{16}^{(2f1p)}$ $D_{26}^{(1f2p)}, D_{26}^{(2f1p)}$ $R_{44}^{(2f2p)}, R_{45}^{(2f1p)}$ $R_{54}^{(1f2p)}, R_{55}^{(1f1p)}$

7.3. Analytical solutions for the special cases

One can be sure that the boundary conditions (88) for the case of hinged immovable support of the shell expansions (89) and the load (90) will be really satisfied.

As to the kinematic conditions it is obvious, and the static conditions must be presented in the expanded form accordingly to the expressions for the forces and moments (72)–(75). Comparing these expressions with Table 1 we can see that the static conditions will be satisfied if their expressions hold the terms with the stiffness characteristics which correspond to the case of angle-ply laminates.

As the conditions (88) for the hinged immovable support are satisfied with expansions (89), (90), for the case of angle-ply laminates, admittedly, the system of differential equations (76) for this case will be also satisfied using these expansions. In order to be sure in this we will use Table 1 for the transformation of the general system (76) to the case of angle-ply laminates. Then we substitute the expansions (89) and (90) into the obtained special system. After substitution we can see that some terms, namely, containing the curvatures of bending, impede the cancellation of trigonometric multipliers. That's why we assume $k_{11} = k_{22} = 0$.

Therewith, the curvature k_{12} does not impede the system of governing algebraic equations to be formed. Therefore, in the case of angle-ply laminates the solution for the shells with positive Gauss curvatures $(k_{11}, k_{22} \neq 0)$ cannot be obtained. However, the solution exists for the shells with negative Gauss curvature $(k_{12} \neq 0, k_{11} = k_{22} = 0)$. This type of shells has a wide application in many engineering branches. Solution for the plates $(k_{11} = k_{22} = k_{12} = 0)$ can be also obtained as a special case.

Thus we obtain the system of linear algebraic equations for the given couples of parameters m, n in the following general form

$$[D_{mn}]\{U_{mn}\} = [D_{Xmn}]\{X_{mn}\} + [F_{mn}]\{P_{mn}\} \tag{97}$$

where vector of Fourier coefficients of sought functions is given as follows

$$\{U_{mn}\} = \{U_{imn}; W_{mn}; X_{pmn}\}^{\mathrm{T}}, \quad i = 1,2; \;\; p = 1,2,3,4 \tag{98}$$

and vector of the given functions determined by the load accordingly to Eq. (51) is

$$\{X_{mn}\} = \{X_{pmn}\}^{\mathrm{T}} = \{P^{\mp}_{imn}\}^{\mathrm{T}} \quad i = 1,2; \;\; p = 5,6,7,8 \tag{99}$$

Vector of the directly given loads is

$$\{P_{mn}\} = \{P^{\mp}_{imn}; P_{3mn}\}^{\mathrm{T}}; \quad i = 1,2 \tag{100}$$

Matrices of the coefficients of the system of linear algebraic equations $[D_{mn}]$, $[D_{Xmn}]$, $[F_{mn}]$ can be obtained by transforming matrices of the system of differential equations $[D]$, $[D_X]$, $[F]$, respectively, by substituting the expansions (89) and (90) of the unknown and given functions into this system. In matrix $[F]$ the following new notations are used:

$$a_0 = a^-; \; a_n = a^+; \qquad \psi^{(1)}_{1f}(a_0) = \psi^-_{1f}(a^-); \qquad \psi^{(n)}_{1f}(a_n) = \psi^+_{1f}(a^+).$$

The full system of governing algebraic equations is given in Appendix A, and for the case of the angle-ply laminates the entire system is given in Appendix B.

The boundary conditions (85) for hinged movable support are satisfied by the expansion of the sought functions (86) and load (87). Therewith, the static conditions can be satisfied only when their expressions, written in accordance with the relations (71)–(74), retain the terms containing the stiffness characteristics for the case of cross-ply laminates (Table 1).

The system of differential equations (76) for cross-ply laminates will be also satisfied with the expansions (86) and (87). However, in doing so, it is necessary to cancel in the system some terms containing the curvatures of torsion k_{12}, i.e. it is assumed that $k_{12} = 0$. Thus, in the case of cross-ply laminates we obtain solution for the shells with positive Gauss curvature ($k_{11}, k_{22} \neq 0, k_{12} = 0$), and in a special case for plates, spherical and cylindrical shells.

The original form of the system of governing equations (97), vectors of sought functions (98) and loads (99), (100) is unchanged. The elements of the system for the cross-ply laminates are given in Appendix C.

This system can be simplified in the case of the shell with cross-ply symmetrical structure through the thickness. In this case the distribution functions of the tangential displacements $\psi^{(k)}_{11}$, $\psi^{(k)}_{12}$, $\psi^{(k)}_{23}$, $\psi^{(k)}_{24}$ turn out to be odd and the functions $\psi^{(k)}_{15}$, $\psi^{(k)}_{16}$ are even.

Solution of the system of algebraic equations for the Fourier coefficients of the sought functions, their amplitudes for the given couples of m and n and further summation of the series accordingly to the given law of the load allows to find the unknown functions. Then using formulae (54)–(60) all the components of stress–strain state of the multilayered shell, in the given points on the surface and within the package of the layers, can be found.

The refined values of transverse shear and normal stresses can be calculated using Eqs. (21) and (22), respectively.

8. Some results

Next we consider some analytical solutions and results for the bending problems in the cross-ply and angle-ply laminated systems.

Problem 1. Let us consider the numerical results of the bending problem of plates depending on the fiber orientation in the angle-ply laminates, and with different number of the layers. As an example we use the material data from Savoia and Reddy (1992) for a unidirectional fiber-reinforced composite:

$$\begin{aligned} &E_L = 174.6\ \text{GPa} \qquad E_T = 7\ \text{GPa} \\ &G_{LT} = 3.5\ \text{GPa} \qquad G_{TT} = 1.4\ \text{GPa} \\ &\nu_{LT} = \nu_{TT} = 0.25 \end{aligned} \tag{101}$$

where L and T refer to the fiber direction and transverse direction, respectively. Three sets of laminates are considered here:

(1) two-layer angle ply $(-\theta, +\theta)$ laminates with layers of equal thickness $(h/2)$
(2) four-layer angle ply $(-\theta, +\theta, -\theta, +\theta)$ laminates with layers of equal thickness $(h/4)$
(3) six-layer angle ply $(-\theta, +\theta, -\theta, +\theta, -\theta, +\theta)$ laminates with layers of equal thickness $(h/6)$

Here the angle θ is measured from the positive x_1-axis. Non-dimensionalized deflections and stresses are presented according to the following definitions (Savoia and Reddy, 1992):

$$\begin{aligned} &[\overline{u}_1, \overline{u}_2] = \frac{[u_1, u_2]E_T \times 10^2}{phS^3} \qquad [\overline{u}_3] = \frac{10^2 E_T u_3}{phS^4} \\ &[\overline{\sigma}_{11}, \overline{\sigma}_{22}] = \frac{[\sigma_{11}, \sigma_{22}(a_1/2, a_2/2)]}{pS^2} \qquad \overline{\sigma}_{12} = \frac{\sigma_{12}(0,0)}{pS^2} \\ &\overline{\sigma}_{13} = \frac{\sigma_{13}(0, a_2/2)}{pS} \qquad \overline{\sigma}_{23} = \frac{\sigma_{23}(a_1/2, 0)}{pS} \end{aligned} \tag{102}$$

where $S = a_1/h$. Normal load is distributed according to sinusoidal law and it corresponds to the coefficients $m = n = 1$ in expansion (95) and (89)

$$p_3 = p \sin\frac{\pi x_1}{a_1} \sin\frac{\pi x_2}{a_2} \tag{103}$$

The plates are square $(a_1 = a_2 = a)$ and have side-to-thickness ratio $a/h = 10$. The numerical results are given in Tables 2–4 and compared with exact 3D solution obtained by (Savoia and Reddy, 1992); with the values obtained from the first order shear deformation theory (FSDT) proposed by Reddy and based on the hypothesis of straight line; with the classical theory of plates (CTP).

The proposed rational theory of laminates is denoted as RTL. As seen from the tables, the results obtained using RTL are in good agreement with 3D solution. The discrepancy is about 3% for the stresses. FSDT gives deviation up to 15–20%, and CTP has large errors for the deflections: 1.5–2 times, despite the fact that the considered plates are comparatively thin. The results for the case when $\theta = 0$ are given only in Table 2 since for the rest of the cases they remain the same.

Problem 2. The influence of the fiber orientation in the angle-ply laminates on deflections and stresses $\overline{\sigma}_{11}, \overline{\sigma}_{13}$ for the plates from the previous problem, is shown in Fig. 3a and b.

The results are given for the six-layered plate. The deflections $\overline{w}_{\max}$ and normal stresses $\overline{\sigma}_{\max} = \sigma_{11}$ are given at the center of the plate, and the transverse shear stresses $\overline{\sigma}_{\max} = \sigma_{13}$ in the middle of the sides. All the considered values have their minimum for the angle $\theta = 45°$. This is especially true for the normal stresses which drop almost threefold. The deflections and the transverse shear stresses are not decreasing so rapidly, only 1.3–1.5 times.

Problem 3. Of interest the study of the dependence between the deflections on the one hand and the stresses on the other hand for the given fiber orientation depending on the number of the layers. Table 5 gives such results for a square plate with side-to-thickness ratio $a/h = 10$ and fiber orientation $(-30°/30°)$.

It can be noted that with increase in the number of layers from 2 to 32 the dimensionless deflections $\overline{w}_{\max}$ decrease approximately 1.8 times, and the stresses $\overline{\sigma}_{11}$, $\overline{\sigma}_{12}$ 1.5 times. The results change steeply when

Table 2
Maximum deflections and stresses in a laminated square two-layered anisotropic plate ($a = 10h$)

θ	Theory	$\overline{w}$	$\overline{\sigma}_{11}$	$\overline{\sigma}_{22}$	$\overline{\sigma}_{12}$	$\overline{\sigma}_{13}$	$\overline{\sigma}_{23}$
0	3D	0.6348	0.5671	0.03567	0.02556	0.4222	0.04644
	RTL	0.6371	0.5701	0.03474	0.02575	0.4225	0.04548
	FSDT	0.6383	0.5248	0.03386	0.02463	0.4315	0.04594
	CTP	0.4313	0.5387	0.02667	0.02128	0.4398	0.03766
15	3D	0.8027	0.5633	0.08062	0.07498	0.3418	0.08138
	RTL	0.8037	0.5701	0.08029	0.07594	0.3413	0.08058
	FSDT	0.8072	0.5330	0.07594	0.7445	0.3447	0.08160
	CTP	0.6205	0.5436	0.07276	0.07028	0.3493	0.07703
30	3D	0.8568	0.4204	0.1696	0.1837	0.2699	0.1570
	RTL	0.8493	0.4258	0.1707	0.1874	0.2704	0.1570
	FSDT	0.8584	0.4044	0.1615	0.1799	0.2729	0.1581
	CTP	0.6842	0.4063	0.1614	0.1790	0.2735	0.1575
45	3D	0.8250	0.2594	0.2594	0.2408	0.2154	0.2154
	RTL	0.8136	0.2625	0.2625	0.2452	0.2155	0.2155
	FSDT	0.8284	0.2498	0.2498	0.2336	0.2174	0.2174
	CTP	0.6547	0.2498	0.2498	0.2336	0.2174	0.2174

Table 3
Maximum deflections and stresses in a laminated square four-layered anisotropic plate ($a = 10h$)

θ	Theory	$\overline{w}$	$\overline{\sigma}_{11}$	$\overline{\sigma}_{22}$	$\overline{\sigma}_{12}$	$\overline{\sigma}_{13}$	$\overline{\sigma}_{23}$
15	3D	0.6150	0.4453	0.06153	0.06739	0.4116	0.1074
	RTL	0.6001	0.4381	0.05886	0.07102	0.4149	0.1102
	FSDT	0.5839	0.3977	0.05429	0.06592	0.4228	0.1116
	CTP	0.3999	0.4090	0.05171	0.06158	0.4285	0.1055
30	3D	0.5619	0.2833	0.1115	0.1331	0.3381	0.1995
	RTL	0.5290	0.2760	0.1068	0.1350	0.3447	0.2037
	FSDT	0.4885	0.2413	0.09346	0.1183	0.3543	0.2092
	CTP	0.3144	0.2431	0.09347	0.1174	0.3550	0.2084
45	3D	0.5430	0.1749	0.1749	0.1642	0.2684	0.2684
	RTL	0.5063	0.1708	0.1708	0.1627	0.2745	0.2745
	FSDT	0.4549	0.1458	0.1458	0.1388	0.2841	0.2841
	CTP	0.2813	0.1458	0.1458	0.1388	0.2841	0.2841

Table 4
Maximum deflections and stresses in a laminated square six-layered anisotropic plate ($a = 10h$)

θ	Theory	$\overline{w}$	$\overline{\sigma}_{11}$	$\overline{\sigma}_{22}$	$\overline{\sigma}_{12}$	$\overline{\sigma}_{13}$	$\overline{\sigma}_{23}$
15	3D	0.5781	0.4295	0.05838	0.06887	0.3696	0.09813
	RTL	0.5717	0.4293	0.05683	0.07257	0.3668	0.09739
	FSDT	0.5589	0.3904	0.05252	0.06738	0.3752	0.09885
	CTP	0.3752	0.4021	0.05003	0.06287	0.3808	0.09320
30	3D	0.5087	0.2662	0.1037	0.1290	0.2957	0.1749
	RTL	0.4942	0.2685	0.1031	0.1342	0.2916	0.1724
	FSDT	0.4599	0.2362	0.09079	0.1183	0.2989	0.1764
	CTP	0.2858	0.2380	0.09081	0.1174	0.2996	0.1757
45	3D	0.4890	0.1640	0.1640	0.1548	0.2355	0.2355
	RTL	0.4725	0.1667	0.1667	0.1593	0.2318	0.2318
	FSDT	0.4281	0.1435	0.1435	0.1372	0.2378	0.2378
	CTP	0.2544	0.1435	0.1435	0.1372	0.2378	0.2378

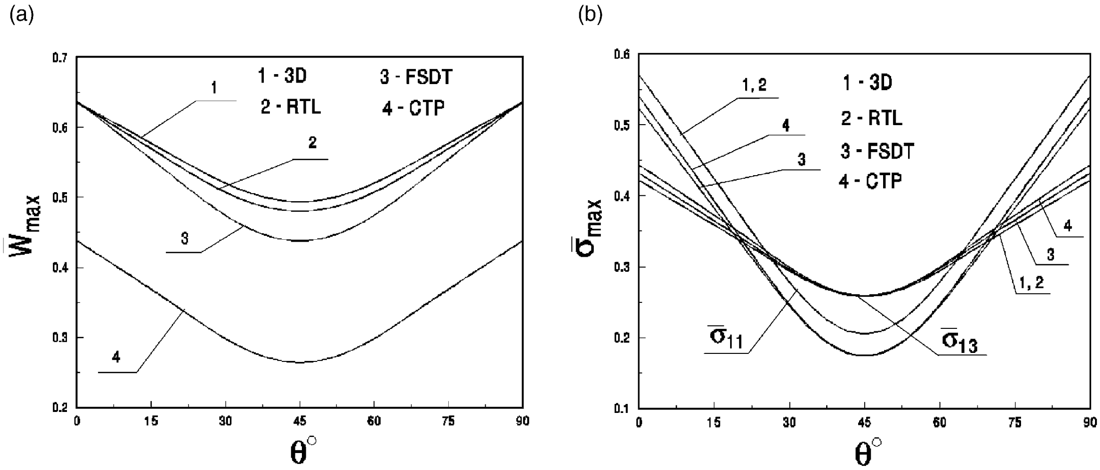

Fig. 3. (a) Displacements in a six-layered angle-ply square plate vs fibre orientation. (b) Stresses in a six-layered angle-ply square plate vs fiber orientation.

Table 5
Maximum displacements and stresses in angle-ply ($\theta = \pm 30°$) laminated square plates ($a = 10h$) with different number of layers

No.	Theory	$\overline{w}$	Δ, %	$\overline{\sigma}_{11}$	Δ, %	$\overline{\sigma}_{12}$	Δ, %	$\overline{\sigma}_{13}$	Δ, %
2	3D	0.857	–	0.420	–	0.184	–	0.270	–
	RTL	0.849	0.9	0.426	1.4	0.187	1.6	0.270	0
	FSDT	0.858	0	0.404	3.8	0.180	2.2	0.273	1.1
	CTP	0.642	25	0.406	3.3	0.179	2.7	0.274	1.5
4	3D	0.562	–	0.283	–	0.133	–	0.338	–
	RTL	0.529	5.9	0.276	2.5	0.136	2.3	0.345	2.1
	FSDT	0.489	13	0.241	15	0.118	11	0.354	4.7
	CTP	0.314	44	0.243	15	0.117	12	0.355	5.0
8	3D	0.490	–	0.264	–	0.130	–	0.303	–
	RTL	0.483	1.4	0.269	1.9	0.135	3.8	0.303	0
	FSDT	0.451	8	0.237	10	0.120	7.7	0.312	3
	CTP	0.277	44	0.239	10	0.119	8.5	0.313	3.3
16	3D	0.473	–	0.268	–	0.135	–	0.294	–
	RTL	0.472	0.2	0.274	2.2	0.140	3.7	0.294	0
	FSDT	0.443	6.3	0.243	9.3	0.124	8.1	0.303	3.1
	CTP	0.269	43	0.244	10	0.123	8.9	0.304	3.4
32	3D	0.468	–	0.274	–	0.139	–	0.292	–
	RTL	0.467	0.2	0.277	1.1	0.141	1.4	0.292	0
	FSDT	0.441	5.8	0.247	10	0.127	8.6	0.298	2.1
	CTP	0.267	43	0.248	10	0.126	9.4	0.299	2.4

Note: the discrepancy with 3D theory is given as an absolute value.

passing from two-layer to four-layer laminates. The deflections decrease almost 1.5 times, and then with increase in the number of layers to 8 they remain approximately at the same level, i.e. the result stabilizes.

The tangential stress also decrease sharply, 1.5 times, when passing from two-layer to four-layer laminates.

Unlike the tangential stresses the transverse shear stresses σ_{13} grow 1.25 times on passing from 2 to 4 layers, and then, with increase in the number of layers above 8 remain at the same level.

Hence the eight-layered system is marginal to stabilize the result.

Decrease of the deflections, which are integral characteristics of the deformed condition, with increase in the number of layers speaks for increase in general stiffness of the system with uniformly distributed material through the thickness starting from eight layers. The system with eight layers and more can be considered as "quasi-homogeneous".

Growth of the general stiffness with increase in the number of layers leads also to lessening of the tangential stresses. However, for the stresses, as for "local characteristics" reaching the "quasi-homogeneous" condition by the system has more significant effect as the "peaks" in the diagram of the stresses smooth out with increase in the number of layers.

Next we consider an interesting phenomena which is essential for accuracy of the calculation results of plates with different numbers of layers (Table 5).

It is observed that two-layered plate has the least error comparing with the exact 3D solution. And FSDT practically gives an exact value for the deflection; RTL has error of the order of 1%; CTP – 25%.

All the theories give much the same values for the stresses. The most errors were obtained for the four-layered laminates. Thus we have for the deflections and normal stresses the following errors, respectively: RTL – 6% and 2.5%; FSDT – 13% and 15%; CTP 44% and 15%. The stabilization of the errors corresponds to the stabilization of the results. The maximum errors for the two-layered plate agree with the results given in Savoia and Reddy (1992).

Problem 4. The analysis of the deflection and stress diagrams for two-layered and four-layered laminates shown in Fig. 4.

The tangential displacements $u_1(0, a_2/2)$ virtually follow the linear distribution law through the thickness. It is in complete agreement with the kinematic model of FSDT, hence the minimum of errors, when using this theory, and also FTL and CTP. The distribution law for the displacements and stresses trough the thickness for the four-layered plate is distinguished by considerable heterogeneity. Computing errors for such structure are the highest.

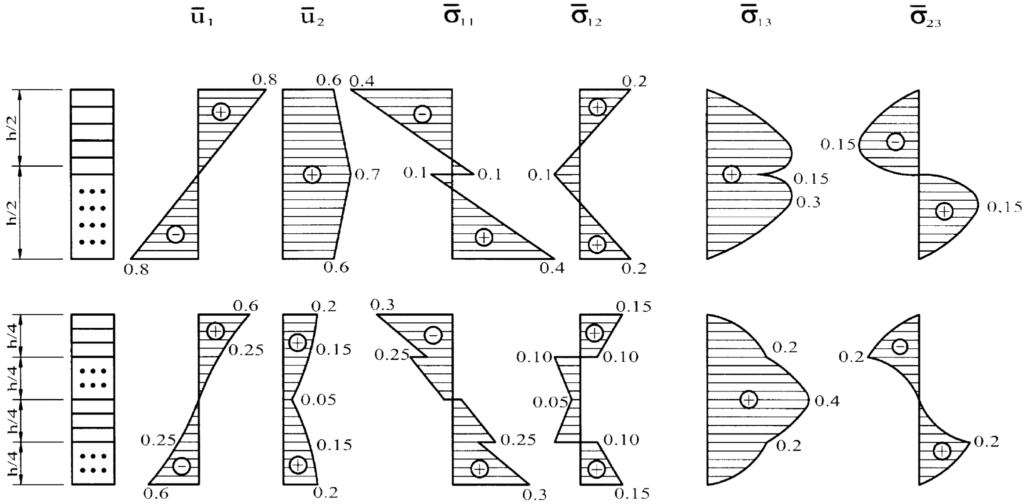

Fig. 4. Displacement and stress diagrams for two- and four-layered laminates.

With the number of layers above 4 the effect of "quasi-homogeneity" is achieved and the errors decrease and stabilize. However, as a whole they do not reach those minimum values which we have for the two-layered system.

Fig. 4 shows general regularities of the stress and displacement distribution. The tangential displacements $\overline{u}_1$ in the middle of each side are asymmetrical through the thickness of the plate in a direction to the edge. The displacements $\overline{u}_2$ are symmetrical in the plane of the plate ends. The tangential normal stresses $\overline{\sigma}_{11}$ have discontinuities on the layer interfaces because of different mechanical characteristics. These and also the tangential shear stresses $\overline{\sigma}_{12}$ are asymmetrical through the thickness. The transverse shear stresses $\overline{\sigma}_{13}$ acting in the plane of the plate ends are symmetrical through the thickness and stresses $\overline{\sigma}_{23}$ are asymmetrical about the normal to the ends.

Symmetry through the thickness of the plate of the displacements $\overline{u}_2$, stresses $\overline{\sigma}_{12}$ and antisymmetry of stresses $\overline{\sigma}_{23}$ are resulted from the asymmetrical structure of the laminates trough the thickness of the plate.

Problem 5. Here an optimal design problem is presented. This is the problem of optimal equal-stress design in angle-ply rectangular plates:

$$\text{optim}\theta(\overline{\sigma}_{11\,\max} = \overline{\sigma}_{22\,\max}, a_1/a_2) \tag{104}$$

This problem is considered for the eight-layered laminated structure. The layers have the equal thicknesses $h/8$. A length of one of the plate sides is constant ($a_2 = 10h$). A length of another side is varying ($a_1 > a_2$). The angle θ is varying too ($0° \leqslant \theta \leqslant 90°$). The layer elastic coefficients are given by (101). The sinusoidal loading is applied to the plate to correspond with (103). The tangential normal stresses $\overline{\sigma}_{11\,\max}$ and $\overline{\sigma}_{22\,\max}$ take place at the center of the plate ($x_1 = a_1/2, x_2 = a_2/2$).

The results of design are shown at the Fig. 5. The curves of the ($\overline{\sigma}_{11\,\max}, \overline{\sigma}_{22\,\max}$) stresses have one point of the intersection for the relative short plates ($a_1/a_2 \leqslant 4$). Two points of the intersections are occurred for long plates ($a_1/a_2 > 4$).

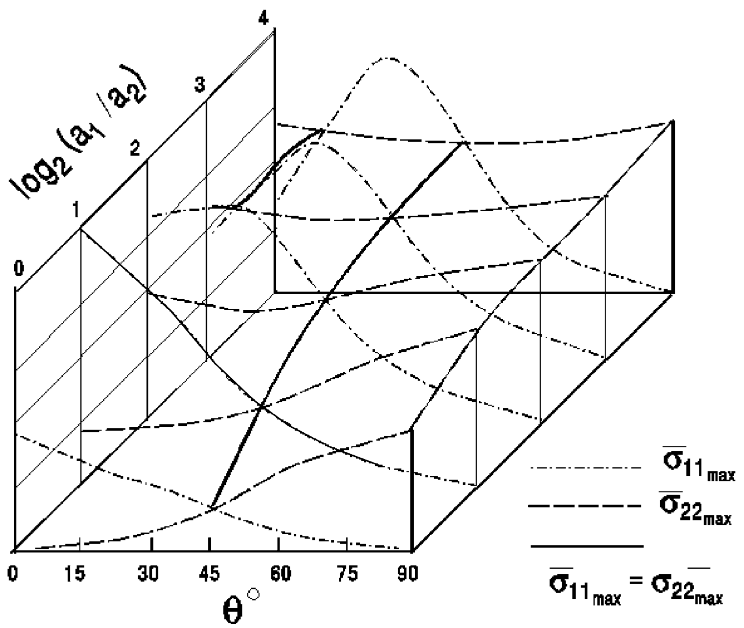

Fig. 5. Optimal equal-stress design of eight-layered angle-ply rectangular plates ($a_2 = 10h$).

The curves connecting points of the equal stresses correspond to the optimal solution. The optimal angles lie within the following bounds: $0° < \theta < 10°$ and $38° < \theta < 43°$ for the long plates; $38° < \theta \leqslant 45°$ for short plates. The equal-stress design gives the possibility to design full-strength laminated structures.

9. Summary and conclusions

A rational higher-order shear deformation theory of anisotropic laminated plates and shallow shells is developed for the solution of static problems subject to both normal and tangential loads. There are three distinctive properties of the proposed theory. First one lies in the fact that it is based on hypotheses which are fully tied to the elastic characteristics of the anisotropic materials of the layers. Secondly, the theory is built on a rational level of difficulty, i.e. it does not add complexity in comparison with known theories of this type and with the theory of orthotropic layered systems. Thirdly, the hypotheses and, correspondingly, all governing relations take into account the influence of the direct application of the tangential loads.

Based on the special approach for the derivation of hypotheses where the boundary conditions are satisfied "step-by-step" on the external surfaces, all the relations of the stress–strain state of the anisotropic laminated shallow shell were obtained. Using the variational approach the equilibrium and the boundary equations are derived. The system of differential equations for the sought functions of displacements and transverse shear is also obtained.

The system takes into account the given shear functions determined by the external tangential loads. The order of the system does not depend on the number of layers in the shell since the theory is of "continuously-structural" type.

The possibility of analytical solution of the system of the governing differential equations in double trigonometric Fourier series is studied. It was determined that in the case of arbitrary shell structure with the anisotropic layers the solution does not exist.

The special cases for which such solution exists are stated: cross-ply laminates for the shallow shells of double positive Gauss curvature with arbitrary stacking sequence; angle-ply laminates for the shallow shells of negative Gauss curvature with asymmetrical lamination through the thickness. These solutions were obtained as special cases of the general solution.

The numerical results for some problems are given. Special attention is given to the angle-ply laminates. A comparison between the proposed RTL theory and exact 3D solution are in very good agreement.

The influence of the fiber orientation of the composite on characteristics of the stress–strain state of plates with different numbers of layers. The parameters were determined for which the stress–strain state and accuracy of the solution stabilizes, for the given overall thickness of the laminated system. A "phenomena" of the high accuracy of the results for the two-layered angle-ply plate is discussed. The behaviour of the variation of the stresses and displacements through the thickness of the asymmetric laminated structure is studied.

The optimal fiber orientation are solved such that the maximum normal stresses are equal, where the aspect ratio of the plate is also design variable and important to design the equal-strength plate structure.

Summarising the results presented in this paper, it is noted that the proposed rational shear deformation theory of the anisotropic laminated shells and plates can lead to a substantial improvement of the accuracy in the following and practical problems:

1. application of numerical methods for the general theory of the anisotropic laminated shells,
2. investigation of the stress–strain state of the anisotropic laminated and, in particular, angle-ply laminated shells of negative Gauss curvature,
3. optimal design of plates and shells with anisotropic layers for practical application in different engineering branches.

Appendix A

The system of governing algebraic equations.

$$\begin{bmatrix} D_{mn}^{(11)} & D_{mn}^{(12)} & D_{mn}^{(13)} & D_{mn}^{(1p)} \\ D_{mn}^{(21)} & D_{mn}^{(22)} & D_{mn}^{(23)} & D_{mn}^{(2p)} \\ D_{mn}^{(31)} & D_{mn}^{(32)} & D_{mn}^{(33)} & D_{mn}^{(3p)} \\ D_{mn}^{(f1)} & D_{mn}^{(f2)} & D_{mn}^{(f3)} & D_{mn}^{(fp)} \end{bmatrix} \begin{Bmatrix} U_{1mn} \\ U_{2mn} \\ W_{mn} \\ X_{pmn} \end{Bmatrix} \qquad p,f = 1,2,3,4$$

$$= \left[\begin{vmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ a^- \lambda_m & a^+ \lambda_m & a^- \gamma_n & a^+ \gamma_n \\ \psi_{1f}^-(a^-) & \psi_{1f}^+(a^+) & \psi_{2f}^-(a^-) & \psi_{2f}^+(a^+) \end{vmatrix} - \begin{vmatrix} D_{mn}^{(1p)} \\ D_{mn}^{(2p)} \\ D_{mn}^{(3p)} \\ D_{mn}^{(fp)} \end{vmatrix} \right] \begin{Bmatrix} P_{1mn}^- \\ P_{1mn}^+ \\ P_{2mn}^- \\ P_{2mn}^+ \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ P_{3mn} \\ 0 \end{Bmatrix}$$

$p = 5,6,7,8;\ \ f = 1,2,3,4$

Appendix B

$$D_{mn}^{(11)} = -(B_{11}\lambda_m^2 + B_{66}\gamma_n^2)$$

$$D_{mn}^{(12)} = D_{mn}^{(21)} = -(B_{12} + B_{66})\lambda_m\gamma_n$$

$$D_{mn}^{(22)} = -(B_{22}\gamma_n^2 + B_{66}\lambda_m^2)$$

$$D_{mn}^{(13)} = D_{mn}^{(31)} = (3C_{16}\lambda_m^2 + C_{26}\gamma_n^2 + 2B_{66}k_{12})\gamma_n$$

$$D_{mn}^{(23)} = D_{mn}^{(32)} = (3C_{26}\gamma_n^2 + C_{16}\lambda_m^2 + 2B_{66}k_{12})\lambda_m$$

$$D_{mn}^{(33)} = D_{11}\lambda_m^4 + D_{22}\gamma_n^4 + 2(D_{12} + 2D_{66})\lambda_m^2\gamma_n^2 + 4k_{12}(C_{16}\lambda_m^2 + C_{26}\gamma_n^2 - 2B_{66}k_{12})$$

$$D_{mn}^{(1p)} = D_{mn}^{(f1)} = (B_{12}^{(2p)} + 2B_{16}^{(1p)} + B_{66}^{(2p)})\lambda_m\gamma_n$$

$$D_{mn}^{(2p)} = D_{mn}^{(f2)} = (B_{22}^{(2p)} + B_{26}^{(1p)})\gamma_n^2 + (B_{16}^{(1p)} + B_{66}^{(2p)})\lambda_m^2$$

$$D_{mn}^{(3p)} = D_{mn}^{(f3)} = -\left[(C_{11}^{(1p)} + C_{16}^{(2p)})\lambda_m^2 + (C_{12}^{(1p)} + 3C_{26}^{(2p)} + 2C_{66}^{(1p)})\gamma_n^2 + 2(B_{16}^{(1p)} + B_{66}^{(2p)})k_{12}\right]\lambda_m$$

$p = 1,2,5,8;\ \ f \rightleftharpoons p = 1,2$

$$D_{mn}^{(1p)} = D_{mn}^{(f1)} = (B_{11}^{(1p)} + B_{16}^{(2p)})\lambda_m^2 + (B_{16}^{(2p)} + B_{66}^{(1p)})\gamma_n^2$$

$$D_{mn}^{(2p)} = D_{mn}^{(f2)} = (B_{12}^{(1p)} + 2B_{26}^{(2p)} + B_{66}^{(1p)})\lambda_m\gamma_n$$

$$D_{mn}^{(3p)} = D_{mn}^{(f3)} = -\left[(C_{12}^{(2p)} + 3C_{16}^{(1p)} + 2C_{66}^{(2p)})\lambda_m^2 + (C_{22}^{(2p)} + C_{26}^{(1p)})\gamma_n^2 + 2(B_{26}^{(2p)} + B_{66}^{(1p)})k_{12}\right]\gamma_n$$

$p = 3,4,6,7;\ \ f \rightleftharpoons p = 3,4$

$$D_{mn}^{(fp)} = -\Big[(D_{11}^{(1f1p)} + D_{16}^{(1f2p)} + D_{16}^{(2f1p)} + D_{66}^{(2f2p)})\lambda_m^2 + (D_{22}^{(2f2p)} + D_{26}^{(2f1p)} + D_{26}^{(1f2p)} + D_{66}^{(1p1f)})\gamma_n^2 + (R_{44}^{(2f2p)} + R_{45}^{(2f1p)} + R_{54}^{(1f2p)} + R_{55}^{(1f1p)})\Big]$$

$p = 1,2,5,8;\ \ f = 1,2;$ or $p = 3,4,6,7;\ \ f = 3,4.$

$$D_{mn}^{(fp)} = -\left[(D_{12}^{(1f2p)} + 2D_{16}^{(1p1f)} + D_{66}^{(1f2p)}) + (D_{12}^{(2f1p)} + 2D_{26}^{(2f2p)} + D_{66}^{(2f1p)})\right]\lambda_m\gamma_n \quad p = 3,4,6,7;\ \ f = 1,2$$

$$D_{mn}^{(fp)} = -\left[(D_{12}^{(1p2f)} + 2D_{16}^{(1p1f)} + D_{66}^{(1p2f)}) + (D_{12}^{(2p1f)} + 2D_{26}^{(2p2f)} + D_{66}^{(2p1f)})\right]\lambda_m\gamma_n \quad p = 1,2,5,8;\ \ f = 3,4$$

Appendix C

$$D_{mn}^{(11)} = -(B_{11}\lambda_m^2 + B_{66}\gamma_n^2)$$

$$D_{mn}^{(12)} = D_{mn}^{(21)} = -(B_{12} + B_{66})\lambda_m\gamma_n$$

$$D_{mn}^{(22)} = -(B_{22}\gamma_n^2 + B_{66}\lambda_m^2)$$

$$D_{mn}^{(13)} = D_{mn}^{(31)} = \left[C_{11}\lambda_m^2 + (C_{12} + 2C_{66})\gamma_n^2 + (B_{11}k_{11} + B_{12}k_{22})\right]\lambda_m$$

$$D_{mn}^{(23)} = D_{mn}^{(32)} = \left[C_{22}\gamma_n^2 + (C_{12} + 2C_{66})\lambda_m^2 + (B_{22}k_{22} + B_{12}k_{11})\right]\gamma_n$$

$$D_{mn}^{(33)} = D_{11}\lambda_m^4 + D_{22}\gamma_n^4 + 2(D_{12} + 2D_{66})\lambda_m^2\gamma_n^2 + 2\left[(C_{11}k_{11} + C_{12}k_{22})\lambda_m^2 + (C_{22}k_{22} + C_{12}k_{11})\gamma_n^2\right] \\ - 2[(B_{11}k_{11} + B_{22}k_{22})k_{11} + (B_{22}k_{22} + B_{11}k_{11})k_{22}]$$

$$D_{mn}^{(1p)} = D_{mn}^{(f1)} = (B_{11}^{(1p)}\lambda_m^2 + B_{66}^{(1p)})\gamma_n^2$$

$$D_{mn}^{(2p)} = D_{mn}^{(f2)} = (B_{12}^{(1p)} + B_{66}^{(1p)})\lambda_m\gamma_n$$

$$D_{mn}^{(3p)} = D_{mn}^{(f3)} = -\left[C_{11}^{(1p)}\lambda_m^2 + (C_{12}^{(1p)} + 2C_{66}^{(1p)})\gamma_n^2 + (B_{11}^{(1p)}k_{11} + B_{12}^{(1p)}k_{22})\right]\lambda_m \quad p = 1,2,5,6; \; f \rightleftharpoons p = 1,2$$

$$D_{mn}^{(1p)} = D_{mn}^{(f1)} = (B_{12}^{(2p)} + B_{66}^{(2p)})\lambda_m\gamma_n$$

$$D_{mn}^{(2p)} = D_{mn}^{(f2)} = (B_{22}^{(2p)}\gamma_n^2 + B_{66}^{(2p)}\lambda_m^2)$$

$$D_{mn}^{(3p)} = D_{mn}^{(f3)} = -\left[(C_{12}^{(2p)} + 2C_{66}^{(2p)})\lambda_m^2 + C_{22}^{(2p)}\gamma_n^2 + B_{12}^{(2p)}k_{11} + B_{22}^{(2p)}k_{22}\right]\gamma_n \quad p = 3,4,7,8; \; f \rightleftharpoons p = 3,4$$

$$D_{mn}^{(fp)} = -\left[(D_{11}^{(1p1f)}\lambda_m^2 + D_{66}^{(1p1f)}\gamma_n^2) + R_{55}^{(1f1p)})\right] \quad p = 1,2,5,6; \; f = 1,2$$

$$D_{mn}^{(fp)} = -\left[(D_{22}^{(2f2p)}\gamma_n^2 + D_{66}^{(2f2p)}\lambda_m^2) + R_{44}^{(2f2p)}\right] \quad p = 3,4,7,8; \; f = 3,4$$

$$D_{mn}^{(fp)} = -(D_{12}^{(1f2p)} + D_{66}^{(1f2p)})\lambda_m\gamma_n; \quad p = 3,4,7,8; \; f = 1,2$$

$$D_{mn}^{(fp)} = -(D_{12}^{(1p2f)} + D_{66}^{(1p2f)})\lambda_m\gamma_n; \quad p = 1,2,5,6; \; f = 3,4$$

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